University of Colorado, Department of Physics  
PHYS3320, Fall 12, HW 1  
due Wed, Sep 5, in class (no late HWs will be accepted)

On all homework assignments this term, please show your work and explain your reasoning. We try to grade for clarity of explanation as much as we do for mere "correctness of final answer". Feel free to use any notes or texts, or talk to other students, or come to the homework sessions. In the end, all your work should be your own.

1. Suppose the vector potential in a region of space is given by:

\[ A = A_0 \exp \left( -\frac{x^2 + y^2}{2a^2} \right) \hat{z} \]  

(1)

a) What are the units of the given constants \( a \) and \( A_0 \)?

b) Determine the magnetic field from this vector potential (via \( B = \nabla \times A \)) and then determine the current density \( J \) from the magnetic field (using the appropriate Maxwell equation).

(Eq. (1) is given in Cartesian coordinates, but you may use another coordinate system, if that would make the calculation easier. Briefly, but clearly justify your choice.)

c) Separately sketch the vector field \( A \), the magnetic field \( B \) and the current density \( J \) you have found (using any representation you feel conveys the most useful information). Briefly, for each of the plots use words to describe what they look like. In particular, if your representation "hides" information, state what it is.

(I encourage you to use e.g. Mathematica to plot these fields - preferably the one(s) you feel is (are) hard to visualize.)

d) Integrate the current density to mathematically show that the total current flowing through any infinite plane parallel to the \( x-y \) plane is zero. Then, give an argument (without doing any formal integral) why you could have known before calculating that this must be the case.

e) Calculate the divergence of \( J \). What does the value of the divergence imply, in terms of Griffiths, Eq. (5.29) ?

f) Does this set of \( A - B - J \) fields strike you as an unphysical mathematical excercise, or can you imagine some physical system or electronic device which might at least approximately described/represented in this problem. Briefly, discuss.

2. a) Prove that the current density is given by \( J = nqv \), where \( n \) is the number of charge carriers per volume, \( q \) is the charge of each carriers, and \( v \) is the average velocity of each carrier, which is also called the drift velocity.

b) In most metals, there is one conduction electron per atom, and the distance between adjacent atoms is about 0.2 nm. Consider a metal wire with a 1 mm diameter, carrying a current of 10 A (these numbers are typical of the wires in the walls of your home). Compute the drift velocity of the electrons in this wire.

c) Compare your results with the average thermal velocity of an electron at room temperature.
3. The region between two concentric metal spherical spheres (with radius $a$ and $b$, respectively) is filled with a weakly conducting material of conductivity $\sigma$. Assume that the outer shell is electrically grounded, and a battery maintains a potential difference of $|V| = V_0$ between the two shells.

(\textit{In this problem, don't confuse the conductivity $\sigma$ with the surface charge density. Also, for this problem, ignore any dielectric properties of this weakly conducting material.})

a) What total current $I$ flows between the shells? What is the total resistance $R$ of the weakly conducting material between the shells?

b) Suppose the battery would be suddenly disconnected at $t = 0$. Thus, at $t = 0$ the voltage difference between the shells is $V_0$, but there is no battery to maintain this any more. Describe qualitatively what you expect happens over time. Determine the net charge on the shells as a function of $t$ in terms of the resistance $R$ and capacitance $C$. Then, calculate the voltage, and the current that flows between the two shells, i.e. find $V(t)$ and $I(t)$. Does your result agree with your qualitative prediction? Discuss whether/how your answer depends on the specific (spherical) geometry of this situation.

c) For the situation of part b), calculate the heat delivered (lost) to the resistance. Check that your result is equal to the total energy originally stored in the capacitor.

d) Going back to the original setup: Adapt your equation for the resistance in part a) to the situation where a conducting sphere of radius $a$ is embedded in a large uniform volume with conductivity $\sigma$, and held at a potential of $V_0$ with respect to some boundary very far away. What would be the resistance for this arrangement?

e) Now use your result from part d) in the following real-world application: Take a single spherical conductor of radius $a_1$ and lower it by a conducting wire into a large, deep body of water. Do the same with a second conducting sphere of radius $a_2$, separated from the first conductor by a distance $d$ with $d \gg \max(a_1, a_2)$. Set up a fixed potential difference $|V| = V_0$ between the two spheres, perhaps with a car battery. Describe what electrical quantity (quantities) you would measure to determine the resistivity $\rho$ (or, the conductivity $\sigma$) of the water in which these two spheres are immersed. Include an explicit formula telling how you would deduce $\rho$ or $\sigma$ from what you have measured? Insert plausible numbers, do you think this experiment could work?

4) Consider a parallel-plate capacitor attached to a battery of constant voltage $V_0$. The plates are separated by a distance $d$ and have a square area $L^2$, where $L \gg d$ so that edge effects are negligible. The space between the plates is filled with a weakly conducting material that has a non-constant conductivity $\sigma(x) = \sigma_0 + \sigma' x$, where $\sigma_0$ and $\sigma' = \frac{d\sigma}{dx} > 0$ are constants.

a) Argue that the current density $\mathbf{J}$ is independent of position $x$ in between the plates. Then, solve for the electric field between the plates in terms of the battery voltage $V_0$ and the other known quantities. Make a qualitative sketch of the magnitude of the field vs. position $x$.

b) Derive an expression for the charge density $\rho$ between the plates in terms of the known quantities in the problem. Check your answer by considering the limit $\sigma' \rightarrow 0$, i.e. the limit of constant conductivity. Explain briefly.