P3320 LECTURE 8

Maxwell stress tensor: $\overleftrightarrow{T}$

$$T_{ij} = \frac{1}{4\pi}\left[E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}\left(E^2 + B^2\right)\right]$$

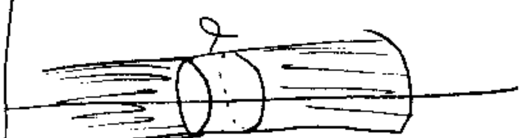
Total force on an object is $\vec{F} = \oint_S \overleftrightarrow{T} \cdot d\vec{A}$

Example: long, uniformly charged cylinder, radius a, charge density ρ. What is the stress tensor?

$$\vec{E} = 2\pi r \rho\, \vec{e}_r$$

$$= 2\pi\rho \sqrt{x^2+y^2}\, \frac{(\vec{e}_x x + \vec{e}_y y)}{\sqrt{x^2+y^2}}$$

$$= \quad (\vec{e}_x x + \vec{e}_y y)$$



$Q_{enc} = \pi r^2 \ell \rho$, $\int \vec{E} \cdot d\vec{A} = 2\pi r E \ell$

$\Rightarrow E = 2\pi r \rho$

So: $$T_{xx} = \frac{1}{4\pi}\left[4\pi^2\rho^2 x^2 - \frac{4\pi^2(x^2+y^2)\rho^2}{2}\right]$$

$$= \pi\rho^2\left(\frac{1}{2}x^2 - \frac{1}{2}y^2\right) = \frac{\pi\rho^2}{2}\left(x^2 - y^2\right)$$

$$T_{yy} = -\frac{\pi\rho^2}{2}\left(x^2 - y^2\right)$$

$$T_{zz} = \frac{-E^2}{8\pi} = \frac{-\pi\rho^2}{2}\left(x^2 + y^2\right)$$

$$T_{xy} = T_{yx} = \frac{E_x E_y}{4\pi} = \frac{4\pi^2\rho^2}{4\pi} xy$$

$$T_{xz} = T_{zx} = T_{yz} = T_{zy} = 0.$$

$$\overleftrightarrow{T} = \pi\rho^2 \begin{pmatrix} \frac{x^2-y^2}{2} & xy & 0 \\ xy & \frac{y^2-x^2}{2} & 0 \\ 0 & 0 & \frac{-(x^2+y^2)}{2} \end{pmatrix}$$

Now, use cylindrical components:

$$T_{rr} = \frac{1}{4\pi}\left(E_r^2 - \frac{1}{2}E^2\right) = \frac{1}{8\pi}(2\pi r\rho)^2 = \frac{\pi r^2 \rho^2}{2}$$

$$T_{zz} = T_{\phi\phi} = \frac{1}{4\pi}\left(-\frac{1}{2}E^2\right) = \frac{-\pi r^2 \rho}{2}$$

All off-diagonals vanish. Can do this in general by aligning one axis with $\vec{E}$ (or $\vec{B}$).

[Another property: Trace of $\mathbb{T}$ is $-\frac{1}{8\pi}(E^2+B^2) = -\left(\substack{\text{energy} \\ \text{density}}\right)$]

Next homework, you'll find the force between opposing halves of the cylinder. Can you use diagonalized tensor?

Momentum and angular momentum:

Lots of "paradox" problems exist where electromagnetic systems appear to violate conservation of momentum or angular momentum. Had similar problem with conservation of energy last semester — fixed it by finding that the field has energy. Will do similarly for momentum: Since we have general expression for force in terms of fields, interpret $\vec{F}$ as $\frac{d\vec{P}}{dt}$, and then:

$$\frac{d\vec{p}}{dt} = \int_V d^3r \left(\operatorname{div}\mathbb{T} - \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t}\right)$$

Does this conserve momentum? Integrate the forces ($\operatorname{div}\mathbb{T}$) and you don't get $\frac{d\vec{p}}{dt}$. So need to consider $\vec{p}_{field}$:

$$\frac{d\vec{p}_{matter}}{dt} + \int_V \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t} = \int_V d^3\vec{r}\ \mathrm{div}\,\mathbb{T}$$

$\Longrightarrow$ integral of stresses $= \frac{d}{dt}\left(p_{matter} + p_{field}\right)$

Introduce momentum density $\vec{p} = \int d^3r\ \vec{g}(\vec{r})$, and remove integral above.

$\vec{p}_{matter} = \int d^3r\ \vec{g}_{matter}$, etc, so

$$\frac{d\vec{g}_{matter}}{dt} + \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t} = \mathrm{div}\,\mathbb{T}$$

$$\Longrightarrow \boxed{\frac{\vec{S}}{c^2} = \vec{g}_{field}}$$

Also can be seen by analogy with Poynting theorem:

$$\underbrace{\frac{1}{8\pi}\frac{\partial}{\partial t}\left(E^2+B^2\right)}_{E_{Field}} + \underbrace{\vec{E}\cdot\vec{J}_f}_{E_{mech}} - \underbrace{\mathrm{div}\vec{S}}_{\text{flux of total energy}} = 0$$

$$\underbrace{\frac{1}{c^2}\frac{\partial \vec{S}}{\partial t}}_{\vec{g}_{field}} + \underbrace{\frac{d\vec{g}_{matter}}{dt}}_{\vec{g}_{matter}} - \underbrace{\mathrm{div}\,\mathbb{T}}_{\substack{\text{flux of total momentum} \\ \text{(i.e. force density)}}} = 0$$