P3220 LECTURE 7

Poynting's theorem: (linear materials)

$$0 = \frac{\partial}{\partial t}\left[\frac{1}{8\pi}\left(\varepsilon E^2 + \mu H^2\right)\right] + \vec{E}\cdot\vec{J}_f + \operatorname{div}\vec{S}$$

Note – found the factor of 2 problem from last lecture: going from $\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}$ to $\frac{\partial}{\partial t}\left(\varepsilon E^2 + \frac{B^2}{\mu}\right)$: factor of 2 from chain rule! $\frac{\partial}{\partial t}B^2 = 2B\frac{\partial B}{\partial t} = 2\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}$

Terms have physical interpretation:

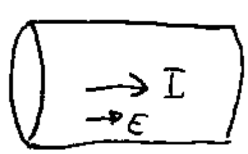
$\frac{1}{8\pi}\left(E^2 + B^2\right)$ is the energy density of the fields

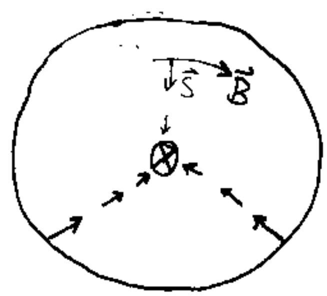
$\vec{E}\cdot\vec{J}_f$ is the mechanical work density

$\Rightarrow$ $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$ is the flux of total energy:

$$\oint d\vec{A}\cdot\vec{S} = -\left(\frac{dW_{mech}}{dt} + \frac{dW_{field}}{dt}\right)$$

Example: (a little counterintuitive): current in a resistive wire.





What if current reversed?

$\vec{E}, \vec{B}$ reverse; $\vec{S}$ does not.

Maxwell Stress-energy tensor: Relate force density to fields in a convenient package. What form can it have?

→ Vector field? Not sufficient, since force on a charge depends on charge's position and velocity.

⟹ Tensor is needed.

Start with force law $\vec{F} = Q\left(\vec{E} + \frac{\vec{u}}{c}\times\vec{B}\right)$

Apply it to a continuous charge distribution:

$$\vec{F} \to d\vec{F} \qquad Q \to \rho\, d^3r \qquad \rho\vec{u} \to \vec{J}$$

So $\vec{F} = \int_V d\vec{F} = \int_V d^3r \left(\rho\vec{E} + \frac{1}{c}\vec{J}\times\vec{B}\right)$ Total force on all charge in the volume.

Substitute $\vec{J} = \frac{c}{4\pi}\left(\text{curl}\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)$, $\rho = \frac{1}{4\pi}\text{div}\vec{E}$;

$$\vec{F} = \int_V d^3r\ \frac{1}{4\pi}\left[(\text{div}\vec{E})\vec{E} + \left(\text{curl}\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}\right]$$

Take time derivative term: $\frac{\partial\vec{E}}{\partial t}\times B$ and rewrite:

$$\frac{\partial}{\partial t}(E\times B) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} - \frac{\partial\vec{B}}{\partial t}\times\vec{E}$$

$$4\pi\frac{d\vec{F}}{d^3r} = (\text{div}\vec{E})\vec{E} + (\text{curl}\vec{B})\times\vec{B} - \frac{1}{c}\left[\frac{\partial\vec{B}}{\partial t}\times\vec{E} + \frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)\right]$$

Now use $\frac{\partial\vec{B}}{\partial t} = -c\,\text{curl}\,\vec{E}$ to make look symmetric with second term.

$$4\pi \frac{d\vec{F}}{d^3r} = (\mathrm{div}\vec{E})\vec{E} + (\mathrm{curl}\vec{B})\times\vec{B} + (\mathrm{curl}\,\vec{E})\times\vec{E} - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

Add a term $(\mathrm{div}\vec{B})\vec{B}$ (which is zero):

$$\frac{d\vec{F}}{d^3r} + \frac{1}{4\pi c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \underbrace{\frac{1}{4\pi}\left[(\mathrm{div}\vec{E})\vec{E} + (\mathrm{curl}\,\vec{E})\times\vec{E} + (\mathrm{div}\vec{B})\vec{B} + (\mathrm{curl}\,\vec{B})\times\vec{B}\right]}$$

This is the divergence of a tensor!

Define $$T_{ij} = \frac{1}{4\pi}\left[E_iE_j + B_iB_j - \tfrac{1}{2}(E^2+B^2)\delta_{ij} \right]$$

so $$\frac{d\vec{F}}{d^3r} + \frac{1}{4\pi c}\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) = \mathrm{div}\,\mathbb{T}.$$

Note that divergence of tensor is a vector. With $\mathrm{div}\,\mathbb{T} = \vec{\nabla}\cdot\mathbb{T}$, can also (correctly) envision ~~dot~~ product of vector with tensor is a vector:

$$(\vec{V}\cdot\mathbb{T})_i = \sum_j V_j T_{ji}$$

Can now extend divergence theorem to tensors: $\oint_S d\vec{A}\cdot\mathbb{T} = \int_V d^3r\,\mathrm{div}\,\mathbb{T}$

...which is a vector integral. Apply to force on charges in a finite volume:

$$\vec{F} = \int_V d^3r\left[\mathrm{div}\,\mathbb{T} + \frac{1}{4\pi c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})\right] = \oint_S d\vec{A}\cdot\mathbb{T} - \frac{1}{4\pi c}\int_V d^3r\,\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$= \int_V d^3r\left[\mathrm{div}\,\mathbb{T} + \frac{1}{c^2}\frac{\partial\vec{S}}{\partial t}\right] = \oint_S d\vec{A}\cdot\mathbb{T} - \frac{1}{c^2}\int_V d^3r\,\frac{\partial\vec{S}}{\partial t}$$

Static limit: forces determined completely by $\mathbb{T}$ at boundary. Note — no reference to charges this applies to — these can be determined from fields.