Potentials in dynamic EM fields:

$$\vec{B} = \text{curl}\,\vec{A}, \quad \vec{E} = -\text{grad}\,\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\text{Lorentz gauge condition: } \text{div}\,\vec{A} = -\frac{1}{c}\frac{\partial \Phi}{\partial t}.$$

Now obtain equations for potentials alone: start with $\vec{E}$, substitute into 1st Maxwell eqn:

$$\text{div}\left(-\text{grad}\,\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 4\pi\rho$$

$$\boxed{-\left(\nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}\text{div}\vec{A}\right) = 4\pi\rho} \quad *$$

... and into 4th equation $\text{curl}\,\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$:

$$\text{curl}\,(\text{curl}\,\vec{A}) + \frac{1}{c}\frac{\partial}{\partial t}\left(\text{grad}\,\Phi + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = \frac{4\pi}{c}\vec{J}$$

$$\underbrace{\text{curl}\,\text{curl}\,\vec{A}}_{=\text{grad}(\text{div}\vec{A}) - \nabla^2\vec{A}} + \frac{1}{c}\text{grad}\frac{\partial \Phi}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J}$$

so $$\frac{1}{c}\text{grad}\frac{\partial \Phi}{\partial t} + \text{grad}(\text{div}\vec{A}) + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J}$$

or $$\boxed{\text{grad}\left(\text{div}\vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right) + \left(\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2\vec{A}\right) = \frac{4\pi}{c}\vec{J}.} \quad **$$

So $*$ and $**$ are coupled differential equations for potentials as functions of sources. Apply Lorentz condition, and magically:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi\rho$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}$$

D'Alembertian $\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

so $\Box^2\Phi = -4\pi\rho$

$\Box^2\vec{A} = -\frac{4\pi}{c}\vec{J}$

So four 1st-order diff eq's (Maxwell's eqns) for fields become two 2nd order diff. eq's for potentials, and the Lorentz condition can remove the coupling between the Φ and $\vec{A}$ potentials.

These reduce to Poisson's eqn in the static limit:

$$\nabla^2\Phi = -4\pi\rho$$
$$\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{J}$$

So, knowing this is satisfied for $\Phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$,

can find equivalent for $\vec{A}$ by noting that the $\vec{A}$ equation is actually a separate eqn for each component:

$$\nabla^2 A_i = -\frac{4\pi}{c}J_i .$$

So, $A_i(\vec{r}) = \frac{1}{c}\int d^3r' \frac{J_i(\vec{r}')}{|\vec{r}-\vec{r}'|} \Rightarrow \vec{A}(\vec{r}) = \frac{-1}{c}\int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$.

Note: quasistatic condition — can approximate potentials using above if $\frac{\partial^2(\Phi,\vec{A})}{\partial t^2} \approx 0$. Then, can use Coulomb & Biot-Savart to find fields and just add $-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ to $\vec{E}$. In effect, this gives

$$\vec{E} = \int d^3r' \left(\frac{\rho(\vec{r}')}{r^2}\vec{e}_{\vec{r}-\vec{r}'} - \frac{1}{c^2 r}\frac{\partial\vec{J}}{\partial t}\right)$$

Basically quasistatic assumption is calculation of magnetic fields assuming Biot-Savart law: generally OK unless distances are very long, or radiative fields are large. In this limit, can assume $\vec{B}$ needs no $\frac{\partial E}{\partial t}$ correction in Faraday's law.

Energy and the Poynting theorem: want to find a conservation law:

Start with the curl relations [in matter form]:

$$\text{curl}\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\text{curl}\,\vec{H} = \frac{4\pi}{c}\vec{J}_f + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

Dot the first eqn. with $\vec{H}$ and second with $\vec{E}$:

$$\vec{H}\cdot\text{curl}\,\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\frac{1}{2}(\vec{H}\cdot\vec{B})$$

$$\vec{E}\cdot\text{curl}\,\vec{H} = \frac{4\pi}{c}\vec{E}\cdot\vec{J}_f + \frac{1}{c}\frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D})$$

Subtract the equations:

$$\underbrace{\vec{H}\cdot\text{curl}\,\vec{E} - \vec{E}\cdot\text{curl}\,\vec{H}}_{} = \frac{-1}{2c}\frac{\partial}{\partial t}\left[\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D}\right] + \frac{4\pi}{c}\vec{E}\cdot\vec{J}_f$$

Identity: this is $\text{div}(\vec{E}\times\vec{H}) \Rightarrow$ Define $\vec{S} \equiv \frac{c}{4\pi}\vec{E}\times\vec{H}$

Poynting vector.

$$\text{div}\,\vec{S} = -\left[\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D}\right) + \vec{E}\cdot\vec{J}_f\right]$$ Poynting's Theorem

Integrate using divergence theorem:

$$\oint d\vec{A}\cdot\vec{S} = \int_V \text{div}\,\vec{S}\, d^3r = -\int d^3r \left[\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D}\right) + \vec{E}\cdot\vec{J}_f\right]$$