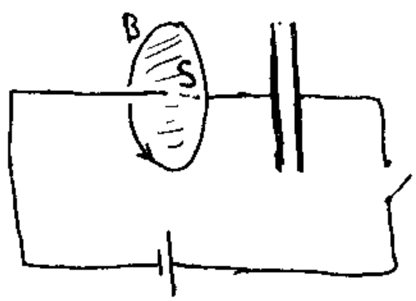
A bit more on displacement current:

$$\text{curl}\,\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

where we can define $\vec{J}_d \equiv \frac{1}{4\pi}\frac{\partial \vec{E}}{\partial t}$ and write

$$\text{curl}\,\vec{B} = \frac{4\pi}{c}\left(\vec{J} + \vec{J}_d\right)$$

A nice way to visualize:

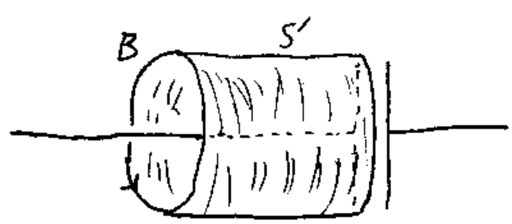


When switch is closed, current $I(t)$ will flow in the circuit, creating $\vec{B}(t)$ around the wire. As this is happening, capacitor will charge up. Continuity (charge conservation) says Q on capacitor obeys $Q(t) = \int_0^t dt'\, I(t')$.

What is effect of this current on magnetic fields? Take surface S, boundary B, around wire, and assume $\vec{E} \simeq 0$ here.

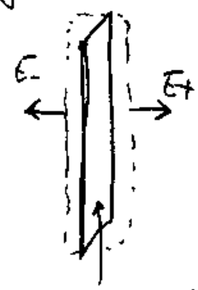
$$\oint_B d\vec{l}\cdot\vec{B} = \frac{4\pi}{c}\int_S \left(d\vec{A}\cdot\vec{J} + \frac{1}{4\pi}\, d\vec{A}\cdot \underbrace{\frac{\partial \vec{E}}{\partial t}}_{0}\right) = \frac{4\pi}{c} I(t).$$

We had better get the same result if we use the same boundary B, but a different surface: S' is a soup can shape with one end open, whose edge is B, and the other end between the capacitor plates.



Now, $I_{encl} = \int_{S'} d\vec{A}\cdot\vec{J} = 0$. What is $\frac{\partial \vec{E}}{\partial t}$?

$\vec{E}$ from surface charge (no fringe fields assumed) can be found by Gauss's Law: for flat surface of S'.



$$\left(\vec{E}_+ + \vec{E}_-\right)A = 4\pi Q\,\hat{n}.$$

Since $\vec{E}_- = 0$,

$$E_+ = \frac{4\pi Q}{A}.$$

Cylindrical surface of S' has $\vec{E} \approx 0$.

So
$$\oint_B d\vec{l}\cdot\vec{B} = \frac{4\pi}{c}\left[\int_{S'}\Big(d\vec{A}\cdot\underbrace{\vec{J}}_{0}\Big) + \int_{S'}\frac{1}{4\pi}d\vec{A}\cdot\frac{\partial \vec{E}_+}{\partial t}\right]$$

$$= \frac{1}{c}\int_{\substack{\text{between}\\ \text{plates}}} d\vec{A}\cdot\frac{\partial E_+(t)}{\partial t}$$

$$= \frac{1}{c}A\cdot\frac{\partial}{\partial t}\left(\frac{4\pi Q}{A}\right) = \frac{4\pi}{c}\frac{\partial Q}{\partial t} = \frac{4\pi I(t)}{c}$$

$\rightarrow$ I_d in the capacitor gap $=$ I in the wire.

Usually, use displacement current

Potentials of time-varying fields:

$\vec{E}$ potential for static fields:

$$\vec{E} = -\text{grad}\,\Phi$$

Can get away with this because $\text{curl}\,\vec{E} = 0$. With time-dependent fields, the scalar potential doesn't work.

What about $\vec{B} = \text{curl}\,\vec{A}$? This works because $\text{div}\,\vec{B} = 0$.

Now, express curl $\vec{E}$ as fn. of $\vec{A}$:

$$\text{curl}\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$= \frac{-1}{c}\frac{\partial}{\partial t}\,\text{curl}\,\vec{A} = -\text{curl}\,\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\Rightarrow \text{curl}\left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 0.$$

Now, can consider the vector $\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ as gradient of a potential. Since $\frac{\partial \vec{A}}{\partial t} = 0$ in static limit, this becomes electrostatic potential:

$$\boxed{-\text{grad}\,\Phi = \vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}}$$

$$\text{or } \vec{E} = -\text{grad}\,\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

What impact has this on our freedom to specify $\vec{A}$ in different gauges? Recall $\vec{B} = \text{curl}\,\vec{A}$ is unchanged under transform $\vec{A}' \rightarrow \vec{A} + \text{grad}\,\xi$ where $\xi(\vec{r})$ is any scalar field.

Under this, $\vec{E} \Rightarrow -\text{grad}\,\Phi - \frac{1}{c}\frac{\partial}{\partial t}(\vec{A} + \text{grad}\,\xi)$

which can be fixed by also changing Φ:

$$\Phi' \rightarrow \Phi - \frac{1}{c}\frac{\partial \xi}{\partial t}$$

(Note that this is only necessary if ξ is time-dependent. Last semester, introduced Coulomb gauge ($\text{div}\,\vec{A} = 0$). This is useful when Laplace equation applies to fields (i.e. no charges).

Lorentz gauge: $\operatorname{div}\vec{A} = -\frac{1}{c}\frac{\partial \Phi}{\partial t}$.

Look at what this does to the potentials:

$$\operatorname{div}\vec{A}' = \operatorname{div}\left(\vec{A} + \operatorname{grad}\xi\right) = -\frac{1}{c}\frac{\partial \Phi}{\partial t}$$

$$\operatorname{div}\vec{A}' = \operatorname{div}\vec{A} + \nabla^2\xi = -\frac{1}{c}\frac{\partial \Phi}{\partial t}$$

Add $\frac{\partial}{\partial t}\left[\Phi' = \Phi - \frac{1}{c}\frac{\partial \xi}{\partial t}\right]$

$$\operatorname{div}A' + \frac{1}{c}\frac{\partial \Phi'}{\partial t} = \operatorname{div}\vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla^2\xi - \frac{1}{c}\frac{\partial^2 \xi}{\partial t^2}$$

$\Longrightarrow$ if all potentials obey Lorentz condition,

$$\nabla^2\xi - \frac{1}{c}\frac{\partial^2 \xi}{\partial t^2} = 0$$

which is a wave equation in ξ.