P3320 LECTURE 42

Let's work the moving-capacitor problem again using the field tensor:

$$\vec{E} = -\gamma_0 4\pi\rho_s \vec{e}_y \,, \quad \vec{B} = -\gamma_0 4\pi\rho_s\beta_0 \vec{e}_z \qquad \text{where } \beta_0 = \frac{u}{c}, \gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}}$$

We know from previous derivation that $E_y' = \gamma(E_y - \beta B_z)$,

$$B_z' = \gamma(B_z - \beta E_y).$$

Field tensor $F^{\mu\nu} \leftrightarrow \begin{pmatrix} 0 & 0 & E_y & 0 \\ 0 & 0 & B_z & 0 \\ -E_y & -B_z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Now $F'^{\mu\nu} = \lambda^\mu{}_\rho F^{\rho\sigma} \lambda_\sigma{}^\nu$ so

$$E_y' = F'^{02} = \lambda^0{}_\rho F^{\rho\sigma} \lambda_\sigma{}^2$$

$$= \lambda^0{}_\rho \Big(F^{\rho 0} \underbrace{\lambda_0{}^2}_{0} + F^{\rho 1} \underbrace{\lambda_1{}^2}_{0} + F^{\rho 2}\lambda_2{}^2 + \underbrace{F^{\rho 3}}_{=0}\lambda_3{}^2 \Big)$$

$$= \lambda^0{}_\rho F^{\rho 2} = \gamma F^{02} - \beta\gamma F^{12} + 0F^{22} + 0F^{32}$$

$$= \gamma E_y - \beta\gamma B_z \quad \checkmark$$

→ You can do B_z' on your own!

Note that the field transformation equations yield the same results if $\vec{E} \longrightarrow \vec{B}$ and $\vec{B} \longrightarrow -\vec{E}$:

$$E_y' = \gamma(E_y - \beta B_z), \quad B_y' = \gamma(B_y + \beta E_z)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$B_y' = \gamma(B_y + \beta E_z), \quad -E_y' = \gamma(-E_y + \beta B_z)$$

If we make this substitution in $F^{\mu\nu}$, we find the dual tensor

$$G^{\mu\nu} \longleftrightarrow \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}$$

... which gives identical fields to $F^{\mu\nu}$ under transformation.

We can now ask what electrodynamics looks like. Start by looking for four-vectors, since force laws will have to be (eventually) expressible in terms of Minkowski force.

Begin with the proper charge density: this is the local charge density in the frame where current vanishes:

$$\rho_0 = \frac{Q}{V_0}$$ where V_0 is the volume in that frame.

Clearly, $\rho = \gamma_u \rho_0$ and current density $\vec{J} = \rho \vec{u} = \rho_0 \gamma_u \vec{u}$

If we take $J^0 = \rho c = \gamma_u \rho_0 c$, then J^μ looks like an invariant (ρ_0) times the local four-velocity.

$\Rightarrow J^\mu$ is the current/charge density four-vector.

Consider the divergence condition on $\vec{E}$: $\operatorname{div}\vec{E} = 4\pi\rho = \frac{4\pi}{c} J^0$

... and the curl condition on $\vec{B}$: $\operatorname{curl}\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

Can write as $\left(\operatorname{curl}\vec{B}\right)_x = \frac{4\pi}{c} J_x + \frac{1}{c}\frac{\partial E_x}{\partial t}$

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \frac{4\pi}{c} J_x + \frac{1}{c}\frac{\partial E_x}{\partial t}$$

or $$\frac{4\pi}{c} J^1 = \frac{-1}{c}\frac{\partial}{\partial t}\left(-F^{10}\right) + \frac{\partial}{\partial x^2} F^{12} - \frac{\partial}{\partial x^3}\left(-F^{13}\right)$$

$$\frac{4\pi}{c} J^1 = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3} = \frac{\partial}{\partial x^\nu} F^{1\nu}$$

Since the div$\vec{E}$ equation is also expressible as

$$\frac{4\pi}{c} J^0 = \frac{\partial F^{0i}}{\partial x^i} + \frac{\partial F^{00}}{\partial x^0} = \frac{\partial}{\partial x^\nu} F^{0\nu}$$

we can generalize and write these two Maxwell equations as

$$\frac{\partial}{\partial x^\nu} F^{\mu\nu} = \frac{4\pi}{c} J^\mu$$

It turns out that the homogeneous equations (the div$\vec{B}$ and curl$\vec{E}$ ones with no source terms) are written as

$$\frac{\partial}{\partial x^\nu} G^{\mu\nu} = 0 .$$

Notation: $\frac{\partial}{\partial x^\nu}$ is usually written as ∂_ν , so:

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} J^\mu$$
$$\partial_\nu G^{\mu\nu} = 0 .$$

An example: An EM wave linearly polarized in X, propagating in Z:

$$\vec{E} = \vec{E}_0 e^{i(kz-\omega t)} , \quad \vec{B} = \vec{B}_0 e^{i(kz-\omega t)} \qquad \vec{S}_0 = \frac{c}{4\pi} \vec{E}_0 \times \vec{B}_0 = \frac{c}{4\pi} E_0^2 \vec{e}_z$$

Where $\vec{E}_0 = E_0 \vec{e}_x$, $\vec{B}_0 = E_0 \vec{e}_y$.

Boost by β in X:

$$\left.\begin{aligned} \vec{E}_0' &= E_0 \vec{e}_x + \beta\gamma E_0 \vec{e}_z \\ \vec{B}_0' &= \gamma E_0 \vec{e}_y \end{aligned}\right\} \vec{S}_0' = \frac{c}{4\pi} \vec{E}_0' \times \vec{B}_0' = \frac{c}{4\pi} E_0^2 \left(\gamma \vec{e}_z - \beta\gamma^2 \vec{e}_x \right)$$