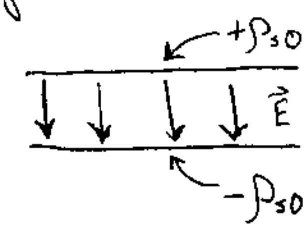
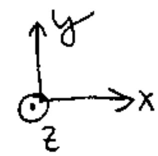
P3320 LECTURE 41

Transforming magnetic fields: Once again, start with a system we can calculate in many frames — the capacitor.





In its rest frame, surface charge density is $\pm\rho_{s0}$.
In the lab frame, the plates are moving at speed u to the right, and the surface charge densities are $\rho_s = \pm\gamma_0\rho_{s0}$ where $\gamma_0 = \dfrac{1}{\sqrt{1-\frac{u^2}{c^2}}}$.

In the lab frame, $\vec{B}$ between the plates is $-\frac{4\pi}{c}\rho_s u\,\vec{e}_z$

Now, introduce a new frame moving at v to the right. What is the velocity of the capacitor in this frame? Use the Einstein velocity addition rule:

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

Now, introduce $\gamma' = \dfrac{1}{\sqrt{1-\frac{u'^2}{c^2}}}$ so $\vec{B}' = -\frac{4\pi}{c}\gamma'\rho_{s0}u'\,\vec{e}_z$

We want to express $\vec{E}'$ and $\vec{B}'$ in terms of $\vec{E}$, $\vec{B}$, and v: this is the general transformation of the field, since it won't refer explicitly to ρ_s and u (which are specific to this geometry).

$$E'_y = -4\pi\gamma'\rho_{s0} \qquad B'_z = -\frac{4\pi}{c}\gamma'\rho_{s0}u'$$

$$E_y = -4\pi\gamma_0\rho_{s0} \qquad B_z = \frac{-4\pi}{c}\gamma_0\rho_{s0}u$$

So $\frac{E_y'}{E_y} = \frac{\gamma'}{\gamma_0} = \frac{\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{1-\frac{uv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\left(1-\frac{uv}{c^2}\right)$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ characterizes the lab frame $\rightarrow$ primed frame boost.

So $E_y' = -\gamma\left(1-\frac{uv}{c^2}\right)4\pi\rho_{s0}\gamma_0 = \gamma\left(E_y - \frac{v}{c}B_z\right)$

$$B_z' = -\gamma\left(1-\frac{uv}{c^2}\right)\frac{4\pi}{c}\left(\frac{u-v}{1-\frac{uv}{c^2}}\right)\rho_{s0}\gamma_0 = -\gamma\left(u-v\right)\gamma_0\frac{4\pi}{c}\rho_{s0} = \gamma\left(B_z - \frac{v}{c}E_y\right).$$

Generalizing to the other components, and taking $\beta = \frac{v}{c}$:

$$E_x' = E_x \qquad E_y' = \gamma\left(E_y - \beta B_z\right) \qquad E_z' = \gamma\left(E_z + \beta B_y\right)$$

$$B_x' = B_x \qquad B_y' = \gamma\left(B_y + \beta E_z\right) \qquad B_z' = \gamma\left(B_z - \beta E_y\right)$$

If one field vanishes in the unprimed frame, things become very simple:

If $\vec{B} = 0$, then:

$B_x' = 0,\ B_y' = \beta\gamma E_z,\ B_z' = -\beta\gamma E_y$

$E_x' = E_x,\ E_y' = \gamma E_y,\ E_z' = \gamma E_z$

so $\vec{B}' = \beta\left(E_z'\vec{e}_y - E_y'\vec{e}_z\right) = -\vec{\beta}\times\vec{E}'$

Similarly, if $\vec{E} = 0$, then $\vec{E}' = +\vec{\beta}\times\vec{B}'$

so $(\vec{\beta}, \vec{B}', \vec{E}')$ form a right-handed system.

This provides a way to transform fields easily if either $\vec{E}$ or $\vec{B}$ vanishes in one frame.

The general field transformations are neatly expressed by applying the $\lambda^\mu{}_\nu$ matrix to a rank-2 antisymmetric tensor describing the EM field:

$$F^{\mu\nu} \Leftrightarrow \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

How to find $F'^{\mu\nu}$: need to contract it <u>twice</u> with λ:

$$F'^{\mu\nu} = \lambda^\mu_\rho \lambda^\nu_\sigma F^{\rho\sigma}$$

In matrix form, $\lambda^\mu_\rho F^{\rho\sigma}$ is left-right multiplication, and $\lambda^\nu_\sigma F^{\rho\sigma}$ is right-left multiplication. So can write:

$$(F') = (\lambda)(F)(\lambda)$$

in matrix notation, <u>remembering the</u> covariant/contravariant sign flips!