P3320 LECTURE 4

Charge continuity: Total charge appears to be conserved precisely. Result: the continuity equation:

$$\operatorname{div} \vec{J} + \frac{\partial \rho}{\partial t} = 0 .$$

In conductor, can combine with Ohm's Law: $\vec{J} = \sigma \vec{E}$

$$\Rightarrow \quad 4\pi\rho\sigma + \frac{\partial \rho}{\partial t} = 0$$

... which is a differential equation for $\rho(t)$, with solution

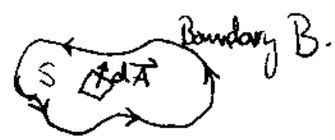
$$\rho = \rho_0 e^{-\frac{t}{\tau}} \qquad \text{with } \tau \equiv \frac{1}{4\pi\sigma} \quad \text{(relaxation time)}.$$

With concept of resistivity, can begin study of <u>electrodynamics</u>.

Start by stating Faraday's Law:

$$\boxed{\operatorname{curl} \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}}$$

We have a curl relation $\rightarrow$ use Stokes's Theorem to make it useful: take a test surface S:



$$\int_S d\vec{A} \cdot \operatorname{curl} \vec{E} = \oint_B d\vec{l} \cdot \vec{E}$$

$$-\frac{1}{c}\int_S d\vec{A} \cdot \frac{\partial \vec{B}}{\partial t} = \oint_B d\vec{l} \cdot \vec{E}$$

Can pull time derivative out of space integral:

$$-\frac{1}{c}\frac{\partial}{\partial t}\underbrace{\int_S \vec{dA}\cdot\vec{B}}_{\equiv \Phi_m \text{ magnetic flux through surface.}} = \oint_B \vec{dl}\cdot\vec{E}$$

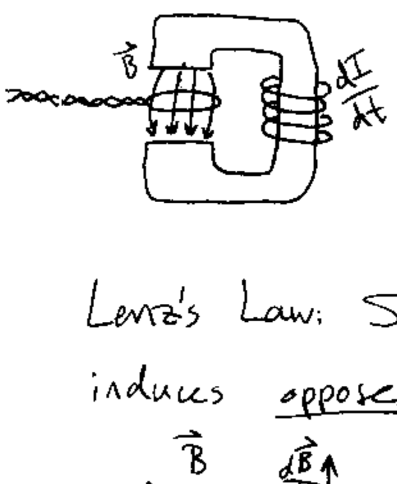
$\oint_B \vec{dl}\cdot\vec{E}$ looks like a potential — but it's around a <u>closed</u> loop! If this were a static field, this integral would have to vanish.

This integral is an electromotive force: EMF. It has potential units and behaves in some ways like one (i.e. it can drive current when maintained in a conductor).

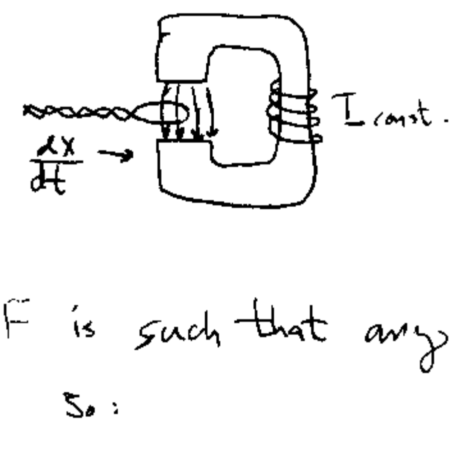
So in integral form, Faraday's Law says $\text{EMF} = -\frac{1}{c}\frac{d\Phi_m}{dt}$.

→ This phenomenon is called <u>induction</u>.

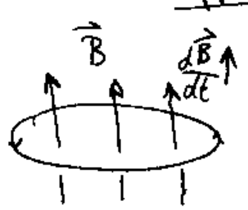
Can generate an inductive EMF either by changing $\vec{B}$ or by moving a loop in and out of a field:



<u>or</u>



Lenz's Law: Sign of EMF is such that any current it induces <u>opposes</u> $\frac{d\Phi_m}{dt}$. So:



if B is growing $\frac{\partial\Phi_m}{\partial t}$ is positive upward, so

EMF is negative clockwise: a <u>negative</u> $\vec{B}_z$ results from this field.

Now, look at what time derivatives do to Ampere's law:

$$\text{curl}\,\vec{B} = \frac{4\pi}{c}\vec{J}$$

Turn it around: $\vec{J} = \frac{c}{4\pi}\,\text{curl}\,\vec{B}$

take divergence: $\text{div}\,\vec{J} = \frac{c}{4\pi}\,\cancel{\text{div}(\text{curl}\,\vec{B})}^{\,0} \Rightarrow \text{div}\,\vec{J} = 0.$

This doesn't seem to agree with continuity eqn $\text{div}\,\vec{J} = -\frac{\partial \rho}{\partial t}$, since there's no law that ρ is constant everywhere!

$\Longrightarrow$ Ampere's law cannot work for time-varying fields!

Work backwards instead:

$$\text{div}\,\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Recall $\rho = \frac{1}{4\pi}\text{div}\,\vec{E}$

$$\text{so} \quad \text{div}\,\vec{J} + \frac{1}{4\pi}\frac{\partial}{\partial t}\text{div}\,\vec{E} = 0$$

Move derivative $\quad \text{div}\left(\vec{J} + \frac{1}{4\pi}\frac{\partial \vec{E}}{\partial t}\right) = 0$

The quantity in parens has zero divergence, so it could be the curl of a vector field. Experimentally, this is the case:

$$\boxed{\text{curl}\,\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}}$$

Maxwell's modification of Ampere's law. This now lets us put all four Maxwell equations together:

$\text{div}\,\vec{E} = 4\pi\rho$ $\qquad\qquad$ $\text{div}\,\vec{D} = 4\pi\rho_f$

$\text{div}\,\vec{B} = 0$ $\qquad\qquad$ in matter:

$\text{curl}\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

$\text{curl}\,\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ $\qquad\qquad$ $\text{curl}\,\vec{H} = \frac{4\pi}{c}\vec{J}_f + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$

Note the clear symmetry btw $\vec{E}$ and $\vec{B}$: in absence of charge, $\operatorname{div}\vec{E} = \operatorname{div}\vec{B} = 0$, $\operatorname{curl}\vec{E} = \frac{-1}{c}\frac{\partial \vec{B}}{\partial t}$ and $\operatorname{curl}\vec{B} = +\frac{1}{c}\frac{\partial \vec{E}}{\partial t}$.

Will soon be very important: electromagnetic waves.

Going back to $\operatorname{curl}\vec{B}$: $\frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

$\frac{1}{4\pi}\frac{\partial \vec{E}}{\partial t}$ plays role of current.

$\Rightarrow$ called "displacement current". It exists if current is blocked in a region of circuit: