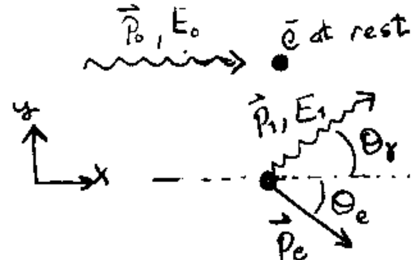
A more complicated example: Compton scattering. A γ-ray (massless, so $p^\mu p_\mu = 0$) scatters off of an atomic electron. The photon energy is huge compared to the electron's kinetic energy, so we can assume the electron is at rest.

Before:

After:



where angles are measured relative to initial γ momentum.

We set the axes so interaction plane is xy.

What is E_1 as a function of θ_γ?

Write out initial 4-momentum: ($m_e \equiv$ electron mass)

$$p^\mu_{init} = \left(\frac{E_0}{c}, p_0, 0, 0\right) + \left(m_e c, 0, 0, 0\right)$$

$$p^\mu_{final} = \left(\frac{E_1}{c}, p_1\cos\theta_\gamma, p_1\sin\theta_\gamma, 0\right) + \left(\frac{E_e}{c}, p_e\cos\theta_e, -p_e\sin\theta_e, 0\right)$$

Know that each component $p^\mu_{init} = p^\mu_{final}$, so:

$$\left.\begin{aligned} &\frac{E_0}{c} + \frac{m_e c^2}{c} = \frac{E_1}{c} + \frac{E_e}{c} \\ \text{or}\quad &E_0 + m_e c^2 = E_1 + \sqrt{m_e^2 c^4 + p_e^2 c^2} \end{aligned}\right\} \text{Total energy conservation}$$

$$p_0 = p_1\cos\theta_\gamma + p_e\cos\theta_e \quad \Big\} \; p^1 \text{ conserved}$$

$$0 = p_1\sin\theta_\gamma - p_e\sin\theta_e \quad \Big\} \; p^2 \text{ conserved.}$$

First, use p^2 equation to eliminate θ_e: $\sin\theta_e = \frac{p_1}{p_e}\sin\theta_\gamma$

But, since photons are massless, $E = pc$ always, so:

$\sin\theta_e = \frac{E_1}{p_e c}\sin\theta_\gamma$ and we can eliminate p_0, p_1.

substitute for $\cos\theta_e$:

Now, $$p_0 = \frac{E_0}{c} = \frac{E_1}{c}\cos\theta_\gamma + p_e\sqrt{1-\left(\frac{E_1}{p_e c}\sin\theta_\gamma\right)^2} \qquad (*)$$

Combine with the 0^{th} component equation
$$E_0 + m_e c^2 = E_1 + \sqrt{m_e^2 c^4 + p_e^2 c^2}$$
$$(E_0 - E_1 + m_e c^2) = m_e^2 c^4 + p_e^2 c^2$$

Eliminate p_e in $(*)$, and after lots of mean, nasty, ugly algebra:
$$E_1 = \frac{E_0}{1 + \frac{E_0}{m_e c^2}(1-\cos\theta)}$$

→ Compton scattering is a dramatic confirmation of both special relativity and the particle nature of high-energy photons.

Forces and acceleration:

Know from problem set that $\vec{F} = m\vec{a}$ is not Lorentz invariant. Actually, it isn't even right (unless you use relativistic mass). Written as
$$\vec{F} = \frac{d\vec{p}}{dt} = \gamma m \frac{d\vec{u}}{dt}$$
it is correct, but note that it's still not Lorentz invariant, nor is it even a good four-vector.

Can create a four-fource (Minkowski force) $K^\mu = \frac{dp^\mu}{d\tau}$ where $K^i = \gamma F^i$. However, when we calculate fields in the lab frame we end up more naturally with $\vec{F}$ than K^μ.

Note that the work-energy theorem still applies:

$$W = \int d\vec{l}\cdot\vec{F} = E_{final} - E_{initial} \quad \text{where } E = \sqrt{p^2c^2 + m^2c^4}.$$

One last mechanics/transformation item: velocity addition rule:
Take a particle moving at u in lab frame, and boost it to a frame moving at v: what is its velocity in primed frame $\frac{dx'}{dt'}$?

$$u = \frac{dx}{dt} \qquad \text{but } dx' = \gamma\left(dx - v\,dt\right)$$

$$dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

$$u' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}} = \boxed{\frac{u - v}{1 - \frac{uv}{c^2}}}$$

(Einstein's velocity addition rule)