Relativistic momentum and energy. Let x^μ be the position of an object that we are studying, in the (unprimed) lab frame.

To define momentum, start by defining velocity.

Is $\frac{dx^\mu}{dt}$ a good four-vector? <u>No</u> — because it's really $\frac{dx^\mu}{dx^0}$ and therefore does not transform like a 4-vector.

Introduce proper time τ $\equiv$ time experienced by the object being analyzed. $\tau = t'_f - t'_i$ if object is not accelerating and we fix the primed frame to the object. What is the invariant interval in this case? $I = (x'_f - x'_i)^2 + (y'_f - y'_i)^2 + (z'_f - z'_i)^2 - c^2(t'_f - t'_i)$ But if the object is fixed in the primed frame, $x_f = x_i$, etc. so $I = -c^2\tau$ and τ is clearly Lorentz-invariant.

Now, try to define four-velocity as $U^\mu \equiv \frac{dx^\mu}{d\tau}$. Numerator is a 4-vector and denominator is invariant $\Longrightarrow$ this velocity is now a good 4-vector! (Note: Griffiths calls four-velocity η^μ)

Properties: first look at $U^0 = \frac{dx^0}{d\tau} = \frac{d(ct)}{d\tau} = c\frac{dt}{d\tau} = \gamma c$

Now, look at the U^i (Roman index goes 1-3 only):

$U^i = \frac{dx^i}{d\tau} = \gamma \frac{dx^i}{dt} = \gamma u^i$ where u^i is the <u>3-velocity</u> of the object in the lab frame $\Rightarrow$ $U^\mu U_\mu = \gamma^2(u^2 - c^2) = \gamma^2 c^2(\beta^2 - 1) = -c^2$

We now form 4-momentum by multiplying by the invariant mass m_0: this is the "rest mass" as defined in H&M.

Now, 4-momentum $p^\mu = m_0 U^\mu$

so its components are $p^i = \gamma m_0 u^i$, $p^0 = \gamma m c$

Two ways to interpret this:

1) "Relativistic mass" $m = \gamma m_0$ so $p^i = m u^i$ is preserved (very old-fashioned)

2) Only use relativistic total energy $E = p^0 c = \gamma m c^2$ so rest energy $E_{rest} = mc^2$ and m is the invariant mass.

We will not speak of relativistic mass again.
Since $E = \gamma m c^2$ and thus rest energy is mc^2, define kinetic energy $T = E - mc^2$.

Now, what is $p^\mu p_\mu$? $\quad p_i p_i - p_0^2 = \gamma m u^2 - \frac{E^2}{c^2}$

(where $u^2 = \vec{u}\cdot\vec{u}$)

$$= \gamma^2 m^2 u^2 - \frac{\gamma^2 m^2 c^4}{c^2}$$

$$p^\mu p_\mu = \gamma^2 m^2 \left(\beta^2 c^2 - c^2\right)$$

$$= -m^2 c^2$$

$$= -\frac{1}{c^2}\left(mc^2\right)^2$$

Generally, more useful to express as $E^2 = p^2c^2 + m^2c^4$
where $p^2 = \vec{p}\cdot\vec{p}$.

Now, introduce a physics law: p^μ is conserved in an isolated system.

We now have all the tools we need to do a wide variety of kinematics problems. Just remember that the four components of p^μ, added for all objects in the system, is the same before and after a collision. This means the invariant mass of the system, $m_{inv} = \frac{\sqrt{E^2 - p^2c^2}}{c^2}$, is conserved, but the sum of the masses of individual components of the system may not be.

A simple example: a particle of mass m_1, momentum $\vec{p}_1$ collides with another of mass m_1 and momentum $-\vec{p}_1$. They stick to each other to form a single object of mass m_2 and momentum $\vec{p}_2$. What are m_2 and $\vec{p}_2$?

start with four-momentum A: $p_A^\mu = \left(\frac{E_A}{c}, p_1^1, p_1^2, p_1^3\right)$

and four-momentum $p_B^\mu = \left(\frac{E_B}{c}, -p_1^1, -p_1^2, -p_1^3\right)$

Add them to find four-momentum of final product:

$$p_2^\mu = \left(\frac{E_A + E_B}{c}, 0, 0, 0, 0\right)$$

What is E_A, E_B? $E_A^2 = p_1^2c^2 + m_1^2c^4$ so $E_2 = 2\sqrt{p_1^2c^2 + m_1^2c^4}$

$E_B^2 = p_1^2c^2 + m_1^2c^4$

so $m_2^2c^4 = E_2^2 - \cancel{p_2^2} = 4\left(p_1^2c^2 + m_1^2c^4\right)$

$$\Rightarrow m_2^2 = 4\left(\frac{p_1^2}{c^2} + m_1^2\right)$$

$$m_2 = 2\sqrt{\frac{p_1^2}{c^2} + m_1^2} \neq 2m_1 \text{ if } \vec{p}_1 \neq 0.$$