P3320 LECTURE 37

Four-vectors and Lorentz transforms:

Define the time-position four-vector to be (ct, x, y, z) and the boost in x. The transforms look more symmetric with ct than t:

We know how these transform:

$$x' = \gamma(x - \beta ct)$$
$$y' = y$$
$$z' = z$$
$$ct' = \gamma(ct - \beta x)$$

If β, γ are treated as parameters, then this is a linear transformation:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Leftrightarrow X'^{\mu} = \Lambda^{\mu}_{\nu} X^{\nu}$$

Note that this can be reversed by reversing the direction of the boost — just switch the sign of β.

Now, one very important property of four-vectors: examine the scalar product of a four-vector with itself.

$$X^{\mu} X_{\mu} = x^2 + y^2 + z^2 - c^2t^2$$

In boosted frame,

$$X'^{\mu} X'_{\mu} = x'^2 + y'^2 + z'^2 - c^2t'^2$$

$$= \gamma^2(x^2 + \beta^2c^2t^2 - 2\beta ctx) + y^2 + z^2 - \gamma^2(c^2t^2 + \beta^2x^2 - 2\beta ctx)$$

$$= y^2 + z^2 + \gamma^2\left[x^2 - c^2t^2 + \beta^2(c^2t^2 - x^2)\right] = y^2 + z^2 + \gamma^2(1-\beta^2)(x^2 - c^2t^2)$$

$$= X^{\mu} X_{\mu}.$$

So all four-vectors' scalar products with themselves are invariant under Lorentz transformations!

We can use this to define the invariant interval between two events. Let the time and place they occur be a^μ and b^μ. Their separation in space and time is $(a-b)^\mu$ which is also a four-vector. The interval $I = (a-b)_\mu (a-b)^\mu = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2$ is calculated in this frame, but is the same calculated in all frame.

Interpretation: If I is negative, then they are connected by a ray moving slower than c. $\Longrightarrow$ there is a frame where they occur at the same location. (the same observer could experience both events.)

I positive: connected by a ray "moving" faster than light. No observer can experience both events, and there is a frame where the events are simultaneous.

Terminology: $I < 0$ timelike interval.
$I > 0$ spacelike interval.

* Note: two events separated by a spacelike interval can have no causality relationship!

(Most intervals between events in everyday life are timelike.)

Length contraction and time dilation:

Take the Lorentz transformation for x coordinate (the boost direction): $x' = \gamma(x - \beta c t)$, and a ruler of length L that is at rest in the x' frame (moving in x frame). Its length at rest is L — so it extends from $x' = 0$ to $x' = L$. What is its length in our (unprimed) frame?

Invert the transform: Left end of ruler is at $0 = \gamma(x_L - \beta ct)$

or $x_L = \beta ct$.

Right end of ruler is at $L = \gamma(x_R - \beta ct)$ or $x_R = \frac{L}{\gamma} + \beta ct$

$\Rightarrow$ Length of ruler is $x_R - x_L = \frac{L}{\gamma}$.

This formula is the source of the famous "ladder paradox."

Time dilation is similar: An observer at rest in the primed frame marks $t'=0$ and $t'=T$, while sitting at $x'=0$.

In unprimed frame, first mark is at

$$ct' = 0 = \gamma(ct - \beta x) \Rightarrow ct = \beta x$$

Second mark is at $ct' = cT = \gamma(ct - \beta x)$

or $ct = \frac{cT}{\gamma} + \beta x$.

so $c\Delta t = \frac{cT}{\gamma}$.

Momentum and energy

We can define a new four-vector describing the momentum of an object. Introduce for explanation purposes (mostly) the four-velocity $u^\mu \equiv \frac{dx^\mu}{d\tau}$ where τ is the proper time (experienced by the moving object). τ is a Lorentz invariant $\left(\tau = \frac{-I}{c}\right)$;

What is $d\tau$? $\frac{1}{c}\sqrt{dx^\mu dx_\mu} = \frac{1}{c}\sqrt{dx^i dx_i - c^2 dt^2}$

$$= dt\sqrt{1 - \frac{1}{c^2}\frac{dx^i dx_i}{dt^2}}$$