Quadrupole radiation: start with superposition of two "physical" dipoles, π out of phase; separated axially by h:

This is an "axial quadrupole"

We now clearly have zero net dipole moment, but nonzero quadrupole moment. Retardation on the scale of $\frac{d}{2}$ will be responsible for any nonzero radiative fields. Without assuming a particular separation distance (except $h \ll r$), we can write the radiation fields:

$$E_\theta = \frac{\ddot{p}(t_{ret})}{c^2 r}\sin\theta \quad \text{from each dipole.}$$

Now, $r - r_+ \approx r_- - r \approx \frac{h}{2}\cos\theta$ and $\ddot{p}(t_{ret}) = -\omega^2 p(t_{ret})$.

At a point $\vec{r}$, then, the radiation fields are approx.

$$E_{\theta_{rad}} = \frac{-\omega^2 p_0}{c^2 r}\sin\theta\, e^{-i\omega t}\left[e^{+\frac{i\omega r_+}{c}} - e^{+\frac{i\omega r_-}{c}}\right]$$

$$\approx \frac{-\omega^2 p_0}{c^2 r}\sin\theta\, e^{-i\omega t} e^{\frac{i\omega r}{c}}\left(-e^{i\omega\frac{h}{2c}\cos\theta} + e^{-i\omega\frac{h}{2c}\cos\theta}\right)$$

Let $t' \equiv t - \frac{r}{c}$

$$= \frac{+\omega^2 p_0}{c^2 r}\sin\theta\, e^{-i\omega t'}\left[2i\sin\left(\frac{\omega h}{2c}\cos\theta\right)\right] = \frac{+2i\omega^2 p_0}{c^2 r}\sin\theta\sin\left(\frac{kh}{2}\cos\theta\right)e^{-i\omega t'}$$

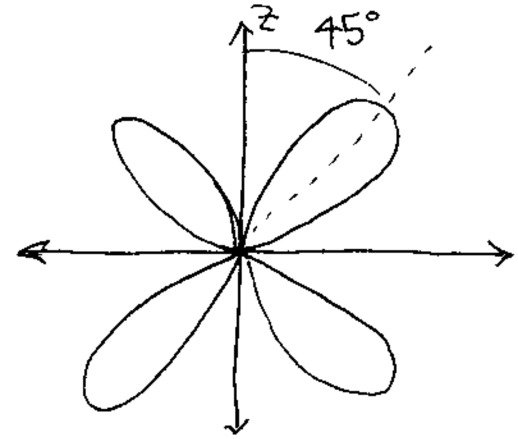
First, consider idealized case: perfect quadrupole ($d \to 0$, $p_0 \to \infty$, keep Q constant). Now, kh is small, so:

$$E_{rad\,\theta} = \frac{+ik\omega^2 p_0 h}{c^2 r}\sin\theta\cos\theta\, e^{-i\omega t'} = \frac{+ik\omega^2 p_0 h}{2c^2 r}\sin 2\theta\, e^{-i\omega t'}$$

and $\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{cr^2}{8\pi} E_{rad}^2 \propto \sin^2 2\theta$.

Radiation pattern:

Maximum radiation at $45°$.



Another useful case is $h = \frac{\lambda}{2}$, or $kh = \pi$. (Note – source is now not $\ll \lambda$)

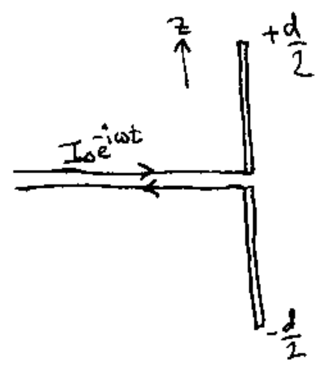
Now, $$E_{\phi\, rad} = +2i \frac{\omega^2 p_0}{c^2 r} \sin\theta \sin\left(\frac{\pi}{2}\cos\theta\right) e^{-i\omega t}$$

and $\left\langle \frac{dP}{d\Omega} \right\rangle \propto \sin^2\theta \sin^2\left(\frac{\pi}{2}\cos\theta\right)$ $= 0$ at $\theta = 0$, $\theta = \pi$, $\theta = \frac{\pi}{2}$.

which actually behaves similarly, but has lobe maximized at $51°$, not $45°$.

Linear antenna is a very similar source:

(Center-driven)



Current is always zero at ends, and is $I = I_0 e^{-i\omega t}$ in center. Assume the gap at center is small, and current varies as a sinusoidal standing wave: $I(z) = I_0 e^{-i\omega t_{ret}} \sin k\left(\frac{d}{2} - |z|\right)$

H&M does the calculation using Jefimenko fields, and find:

$$B_{rad} = -(\vec{e}_z \times \vec{e}_r) \frac{i\omega}{c^2 r} \int dz' \, I(z', t_{ret})$$

and, after finding E_{rad} and $\vec{S}$, gets:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{I_0^2}{2\pi c} \left(\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin\theta} \right)$$

Interesting cases are when d is a half-integer number of wavelengths: $kd = m\pi$. Can consider end-driven case too:

Now can have either a node or antinode in the center. If antinode, looks just like center-driven case, if node (even m), then phases are opposite the center-driven case.

$m=2$ end-driven is very similar to axial quadrupole!

Definition: antenna directivity (gain):

$$G = \frac{\left\langle \frac{dP}{d\Omega} \right\rangle_{max}}{\left\langle \frac{dP}{d\Omega} \right\rangle_{mean}} \qquad \text{where } \left\langle \frac{dP}{d\Omega} \right\rangle_{mean} = \frac{\langle P \rangle}{4\pi}.$$