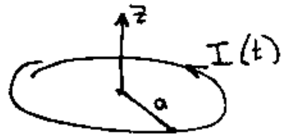
P3320 LECTURE 34

Magnetic dipole radiation: treat a current distribution as a loop of current with area $A = \pi a^2$



(Physically, need a current source)

Magnetic dipole moment is $\vec{m} = \frac{I}{c}\int d\vec{A} = \frac{\pi a^2 I}{c}\vec{e}_z$.

We might expect that the radiated power would be similar to that from the electric dipole $P = \frac{2\ddot{p}^2(t_{ret})}{3c^3} \rightarrow \frac{p_0^2\omega^4}{3c^3}$

and replace $p_0 \rightarrow m_0$.

As with the electric dipole moment, can start by calculating the retarded vector potential for distances $a \ll \lambda \ll r$:

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int d^3r' \frac{\vec{J}(\vec{r}', t_{ret})}{\eta} \quad \text{and } I = I_0\cos\omega t.$$

The scalar potential vanishes, of course, since $\rho = 0$ everywhere.

$$\vec{A} = \frac{1}{c}\oint_{loop} d\vec{l}' \frac{I_0\cos\omega(t-\frac{\eta}{c})}{\eta}$$

We can take $\vec{r}$ such that $\phi = 0$ since the distribution is azimuth-symmetric.

(Note that in H&M, they call the line element $d\vec{l}$ and the ϕ' coord. $\rightarrow \phi$.)

Assume $\frac{1}{\eta} \rightarrow \frac{1}{r}$ in integral, but $t_{ret} \neq t - \frac{r}{c}$ or radiation vanishes.

So $\vec{A} = \frac{I_0}{cr}\oint d\vec{l}' \cos\omega(t - \frac{\eta}{c})$. Now, need to approximate η in the cosine argument.

Can't do the integral in cylindrical coordinates!

$\vec{r} = r\sin\theta\,\vec{e}_x + r\cos\theta\,\vec{e}_z$ (since $\phi = 0$, no $\vec{e}_y$ component)

$\vec{r}' = a\cos\phi'\,\vec{e}_x + a\sin\phi'\,\vec{e}_y$ (since $r' = a$, $\theta' = \frac{\pi}{2}$ on loop)

Now, $\mathscr{r} = |\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'}$

$$= \sqrt{r^2 + a^2 - 2ar\sin\theta\cos\phi'} \overset{\frac{a}{r}\text{ small}}{\approx} r\left[1 - \frac{1}{2}\left(2\frac{a}{r}\sin\theta\cos\phi'\right)\right]$$

So $t_{ret} \cong t - \frac{r}{c} + \frac{a}{c}\sin\theta\cos\phi'$ and integrand becomes

$$\cos\left[\omega\left(t - \frac{r}{c}\right) + \frac{\omega a}{c}\sin\theta\cos\phi'\right]$$

Use angle addition identity:

$$\cos\omega\left(t - \frac{r}{c}\right)\cos\left(\frac{\omega a}{c}\sin\theta\cos\phi'\right) - \sin\omega\left(t - \frac{r}{c}\right)\sin\left(\frac{\omega a}{c}\sin\theta\cos\phi'\right)$$

But $\frac{\omega a}{c}$ is small, so this is:

$$\cong \cos\omega\left(t - \frac{r}{c}\right) - \left[\sin\omega\left(t - \frac{r}{c}\right)\right]\frac{\omega a}{c}\sin\theta\cos\phi'$$

Now do the ϕ' integral: $\vec{d\ell} = a\,d\phi'\left(-\vec{e}_x\sin\phi' + \vec{e}_y\cos\phi'\right)$

$$\vec{A} = \frac{I_0 a}{cr}\int_0^{2\pi} d\phi'\left[-\vec{e}_x\sin\phi'\left(\cos\omega\left(t - \frac{r}{c}\right) - (\text{const})\cos\phi'\right) + \vec{e}_y\cos\phi'\left(\cos\omega\left(t - \frac{r}{c}\right) - (\text{const})\cos\phi'\right)\right]$$

All terms integrate to zero except the $\cos^2\phi'$ term, so

$$\vec{A}(r,\theta,0) = \frac{I_0 a}{cr}\left(\frac{-\omega a}{c}\right)\sin\theta\sin\omega\left(t - \frac{r}{c}\right)\int_0^{2\pi} d\phi'\cos^2\phi'\;\vec{e}_y$$

$$= \frac{-I_0\omega\pi a^2}{c^2 r}\sin\theta\sin\omega\left(t - \frac{r}{c}\right)\vec{e}_y .$$

Now, $\vec{E}_{rad} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} = \frac{\omega^2 I_0\pi a^2\sin\theta}{c^3 r}\cos\left(t - \frac{r}{c}\right)\vec{e}_y$.

At $\phi = 0$, $\vec{e}_y = \vec{e}_\phi$ so we can generalize to all ϕ.

Now, $\vec{B}_{rad} = \vec{E}_{rad} \times \vec{e}_r = \left(\frac{\omega^2 I_0 \pi a^2 \sin\theta}{c^3 r}\right) \cos(t - \frac{r}{c}) \vec{e}_\theta$

So $\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{cr^2}{8\pi} E_{rad\,\theta}^2 = \frac{\omega^4 I_0^2 \pi^2 a^4 \sin^2\theta}{c^6 r^2} \frac{cr^2}{8\pi}$

$$= \frac{I_0^2 \pi^2 a^4 \omega^4 \sin^2\theta}{8\pi c^5} = \frac{m_0^2 \omega^4}{8\pi c^3} \sin^2\theta$$

Integrating for total power $\Rightarrow \frac{m_0^2 \omega^4}{3c^3}$ as expected.

Rad fields are like electric dipole but with $\vec{E}, \vec{B}$ reversed!
(and a - sign on one of them)