P3320 LECTURE 31

Antenna theory / multipole radiation:

Recall potential of an ideal dipole: $\Phi = \frac{p_0 \cos\theta}{r^2}$

where $\vec{p} = p_0 \vec{e}_z$ and dipole is at origin. From "physical" dipole picture:  $\vec{p} = Q\vec{l}$ so $p_0 = Ql$

We can see that if we have the dipole moment changing with time, there will be accelerating charges or a changing current — in either case we expect radiation. Now, naively: simply evaluate Φ at retarded time and see if we can calculate the radiation this way: say $\vec{p} = p_0 \vec{e}_z \cos\omega t$

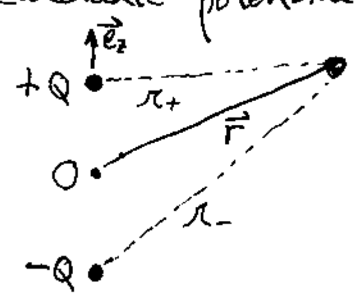
$$\Phi = \frac{p_0 \cos\theta \cos\omega t_{ret}}{r^2} \qquad t_{ret} = t - \frac{\mathscr{r}}{c} = t - \frac{r}{c}$$

since all charges at origin.

Now, $$\Phi = \frac{p_0 \cos\theta \cos\left[\omega\left(t - \frac{r}{c}\right)\right]}{r^2}$$

Ask — can this yield a true radiation field?

Radiation must carry energy to large distances: this means fields have to drop no faster than $\frac{1}{r^2}$: Otherwise integral of $\vec{S}$ will vanish at large radius. But this requires that potentials drop no faster than $\frac{1}{r}$. $\Longrightarrow$ we have approximated too much away. Must return to physical dipole picture to re-calculate potential: set $Q = Q_0 \cos\omega t$, $\vec{l} = d\vec{e}_z$ (fixed):



$$\Phi = \frac{Q_0 \cos\omega t_{ret+}}{\mathscr{r}_+} - \frac{Q_0 \cos\omega t_{ret-}}{\mathscr{r}_-}$$

Make standard multipole approximation: $r, \mathscr{r} >> d$.

Now, can make approximations for $\mathscr{r}_\pm$:

$$\mathscr{r}_\pm = \sqrt{r^2 + \frac{d^2}{4} \mp rd\cos\theta} \approx r \mp \frac{d}{2}\cos\theta \approx r\left(\frac{1}{1 \pm \frac{d}{2r}\cos\theta}\right)$$

Now, evaluate $t_{ret} = t - \frac{\mathscr{r}_\pm}{c} \approx t - \frac{r}{c} \mp \frac{d\cos\theta}{2c}$

so $\cos\omega t_{ret\pm} \cong \cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\left(\frac{\omega d}{2c}\cos\theta\right) \mp \sin\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\left(\frac{\omega d}{2c}\cos\theta\right)$

by angle addition formula.

Now, make another assumption: dipole scale is small compared to wavelength scale: $d << \frac{c}{\omega}$. Now the cos cosθ and sin cosθ terms become ~ 1 and $\sim \frac{\omega d}{2c}\cos\theta$

$$\Phi(\vec{r}, t) = Q_0\left\{\frac{\cos\left[\omega\left(t-\frac{r}{c}\right)\right] - \frac{\omega d}{2c}\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}{\mathscr{r}_+} - \frac{\cos\left[\omega\left(t-\frac{r}{c}\right)\right] + \frac{\omega d}{2c}\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}{\mathscr{r}_-}\right\}$$

Now substitute $\frac{1}{\mathscr{r}_\pm} = \frac{1 \pm \frac{d}{2r}\cos\theta}{r}$

<u>and</u> to do radiation, ignore any term that drops as $\frac{1}{r^2}$ or faster:

$$\Phi(\vec{r}, t) = -\frac{\overbrace{Qd}^{=p_0}\omega}{cr}\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$$

Now, can find $\vec{A}$ by assuming that $I = \frac{dQ(t)}{dt} = -Q_0\omega\sin\omega t$:

$$\vec{A} = -\frac{p_0\omega}{rc}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\vec{e}_z$$

Can go ahead and find fields now:

$$\vec{E} = -\left(\text{grad}\,\Phi + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right), \quad \vec{B} = \text{curl}\,\vec{A}.$$

Start with latter:

$$\vec{B} = \frac{-p_0\omega^2}{c^2 r}\sin\theta\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\vec{e}_\phi$$

But to find <u>radiative</u> $\vec{E}$, need only recall that

$$\vec{E}_{rad} = \vec{B}_{rad} \times \vec{e}_r$$