Potentials and fields from a moving point charge.

We haven't actually covered this, only the potentials of continuous charge and current distributions. We will require certain features for the result of this (not-explicitly-relativistic) calculation:

1. A moving point charge clearly represents a current, though it's very non-constant. $\Longrightarrow$ we expect a nonzero $\vec{A}$ and $\vec{B}$.

2. Special relativity tells us that a point charge moving with constant velocity is at rest in some reference frame $\Longrightarrow$ we do not expect the fields of a constant-velocity point charge to produce radiation.

3. The potential can depend only on the particle's position at one time.

Start with the potential of a continuous distribution:

$$\Phi(\vec{r},t) = \int d^3r' \, \frac{\rho(\vec{r}',t_r)}{\mathscr{r}}$$

Now, take a point charge Q, moving along a path such that its position is $\vec{r}_e(t)$. (Textbook assumes $Q=e$, which is usually the case!) Very tempting to write the potential as simply

$$\Phi(\vec{r},t) = \frac{Q}{|\vec{r}-\vec{r}_e(t_r)|} \qquad (\text{this is wrong!})$$

But — recall that we have to treat the point charge as a limit of a continuous distribution, so $\rho(\vec{r},t) = Q\,\delta^3(\vec{r}_e - \vec{r})$ Now,

$$\Phi(\vec{r},t) = \int d^3r' \, \frac{Q\,\delta^3\{\vec{r}_e(t_{ret}) - \vec{r}'\}}{\mathscr{r}}$$

If t_{ret} were not dependent on $\vec{r}'$, then this would be equal to the "wrong potential."

The dependence yields, effectively, a different integration variable than the one inside the δ — so we'll pick up an integration factor, from changing the integration variable to the variable in the delta fn: $\vec{U} = \vec{r}_e(t_{ret}) - \vec{r}\,'$. Need to evaluate $\frac{d r_i'}{dU_i}$. This is easier if we recognize that the particle is only moving in one direction at a given time, so two dimensions of the δ-fn can be integrated out trivially, leaving a 1-D integral. Define:

$$\vec{\beta}(t) \equiv \frac{1}{c}\frac{d\vec{r}_e}{dt} \quad \text{(velocity in c units.)}$$

Now, integral can be rewritten, integrating components of $\vec{r}\,'$ normal to $\vec{\beta}(t_{ret})$:

$$\Phi(\vec{r},t) = \int dr_\beta' \, \frac{Q\,\delta\left(r_\beta(t_{ret}) - r_\beta'\right)}{\mathcal{R}}$$

Now, switch integration variable to t':

$$\Phi(\vec{r},t) = Q\int_{-\infty}^{\infty} dt' \, \frac{\delta(t_{ret} - t')}{\mathcal{R}}$$

Now, need only integration factor from $t' \to \underbrace{t_{ret} - t'}_{\text{call this } t''}$ substitution:
This turns out to be $\dfrac{|\vec{r} - \vec{r}_e|}{|\vec{r} - \vec{r}_e| - \vec{\beta}\cdot(\vec{r} - \vec{r}_e)}$

or, since $\vec{r}\,' \to \vec{r}_e$ where δ function nonzero, $\dfrac{\mathcal{R}}{\mathcal{R} - \vec{\beta}\cdot\vec{\mathcal{R}}}$.

$$\text{Now, } \Phi(\vec{r},t) = Q\int dt'' \, \frac{\delta(t'')}{\mathcal{R}(t')} \left(\frac{\mathcal{R}(t')}{\mathcal{R}(t') - \vec{\beta}(t')\cdot\vec{\mathcal{R}}(t')} \right)$$

$$= Q\,\frac{1}{\mathcal{R}(t') - \vec{\beta}\cdot\vec{R}}\bigg|_{t''=0} = \frac{Q}{\mathcal{R}(t_{ret}) - \vec{\beta}(t_{ret})\cdot\vec{\mathcal{R}}(t_{ret})}$$

where $\vec{\mathcal{R}}(t_{ret}) \equiv \vec{r}(t) - \vec{r}_e(t_{ret})$.

Similarly for $\vec{A}$, just need another time derivative, so

$$\vec{A}(\vec{r},t) = Q\frac{\vec{\beta}(t_{ret})}{\mathcal{R}(t_{ret}) - \vec{\beta}(t_{ret})\cdot\vec{\mathcal{R}}(t_{ret})} = \beta(t_{ret})\,\Phi(\vec{r},t) \; .$$

These are called the Liénard-Wiechert potentials.

Example: particle on the z axis, with position $\vec{r}_e(t) = vt\vec{e}_z$.
Retarded time is thus determined by

$$|\vec{r} - vt_{ret}\vec{e}_z| = c\,(t - t_{ret})$$

$$\sqrt{x^2+y^2+(z-vt_{ret})^2} = c\,(t - t_{ret})$$

Square: $\underbrace{x^2+y^2+z^2}_{=r^2}+(vt_{ret})^2 - 2zvt_{ret} = c^2t^2 - 2c^2tt_{ret} + c^2t_{ret}^2$

By quadratic formula,

$$t_{ret} = \frac{(c^2t - vz) \pm \sqrt{(c^2t - vz)^2 + (c^2-v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

Which is the correct sign? $t_{ret} \le t$ everywhere, so need the $-$ sign.

Now, $\vec{\mathcal{R}} = \vec{r} - vt_r\vec{e}_z$ and we know $\frac{\mathcal{R}}{t - t_{ret}} = c$ by definition.

So, since we eventually want $\Phi = \frac{Q}{\mathcal{R}(t_{ret}) - \vec{\beta}(t_{ret})\cdot\vec{\mathcal{R}}_{t_{ret}}}$

start with $\mathcal{R} - \vec{\beta}\cdot\vec{\mathcal{R}} = \mathcal{R}(1 - \vec{\beta}\cdot\vec{e}_{\mathcal{R}}) =$

$$c(t - t_{ret})\cdot\left(1 - \frac{v\vec{e}_z}{c}\cdot\frac{(\vec{r} - v\vec{e}_z t_{ret})}{c(t - t_{ret})}\right) = c(t - t_{ret}) - \frac{vz}{c} + \frac{v^2}{c}t_{ret}$$

$$= \frac{1}{c}\left\{(c^2t - vz) - (c^2-v^2)t_{ret}\right\}$$

Now substitute for t_{ret}:

$$= \frac{1}{c}\sqrt{(c^2t - vz)^2 + (c^2-v^2)(r^2 - c^2t^2)}$$

$$\Rightarrow \quad \Phi = \frac{Qc}{\sqrt{(c^2t - vz)^2 + (c^2-v^2)(r^2-c^2t^2)}}$$

$$\vec{A} = \frac{Qv\vec{e}_z}{\sqrt{(c^2t - vz)^2 + (c^2-v^2)(r^2-c^2t^2)}}$$

Finding the fields from this is fairly straightforward:

At $t=0$: $\vec{E} = -\left(\operatorname{grad}\Phi\Big|_{t=0} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\Big|_0\right)$

$$= -Q\left\{\operatorname{grad}\frac{c}{\sqrt{v^2z^2 + (c^2-v^2)r^2}} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\Big|_0\right\}$$

We can write $\Phi(t=0) = \dfrac{Qc}{\sqrt{v^2z^2+(c^2-v^2)r^2}} = \dfrac{Qc}{\sqrt{c^2r^2 - v^2(x^2+y^2)}} = \dfrac{Qc}{rc\sqrt{1-\beta^2\sin^2\theta}}$

where θ is the spherical polar coordinate. Note that for $\beta \to 0$ this reduces to $\Phi = \frac{Q}{r}$.

So $\operatorname{grad}\Phi = Q\operatorname{grad}\dfrac{1}{r\sqrt{1-\beta^2\sin^2\theta}} = Q\left\{\vec{e}_r\left(\dfrac{-1}{r^2\sqrt{1-\beta^2\sin^2\theta}}\right) + \vec{e}_\theta\dfrac{1}{r}\dfrac{\partial\Phi}{\partial\theta}\right\}$

$$= Q\left\{\vec{e}_r\frac{-1}{r^2\sqrt{1-\beta^2\sin^2\theta}} + \frac{\vec{e}_\theta}{r^2}\frac{-1}{(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}\left(\cancel{2}\beta^2\sin\theta\cos\theta\right)\right\}$$

$$\frac{\partial}{\partial t}\vec{A} = Qv\vec{e}_z\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{(c^2t-vz)^2+(c^2-v^2)(r^2-c^2t^2)}}\right)$$

$$= Qv\vec{e}_z\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{\cancel{c^4t^2}-2vzc^2t+v^2z^2+c^2r^2-\cancel{c^4t^2}-v^2r^2+v^2c^2t^2}}\right)$$

$$= Qv\vec{e}_z\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{v^2c^2t^2-2vc^2zt+r^2c^2(1-\beta^2\sin^2\theta)}}\right)$$

$$= Qv\vec{e}_z\left(-\frac{1}{2}\right)\left(\frac{2v^2c^2t-2vc^2z}{\left(v^2c^2t^2-2vc^2zt+r^2c^2(1-\beta^2\sin^2\theta)\right)^{\frac{3}{2}}}\right)$$

set $t=0$:

$$= +Qv\vec{e}_z\,\frac{vc^2z}{r^3c^3(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}$$

But $z = r\cos\theta$

$\vec{e}_z = \vec{e}_r\cos\theta - \vec{e}_\theta\sin\theta$

so

$$\frac{\partial\vec{A}}{\partial t} = +Qv(\vec{e}_r\cos\theta - \vec{e}_\theta\sin\theta)\frac{v\cos\theta}{r^2c(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}$$

$$= +Q\left\{\frac{\beta^2c\cos^2\theta}{r^2(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}\vec{e}_r - \frac{\beta^2c\sin\theta\cos\theta}{r^2(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}\vec{e}_\theta\right\}$$

$$\vec{E} = -\left(\frac{1}{c}\frac{\partial\vec{A}}{\partial t}+\mathrm{grad}\,\Phi\right) \Rightarrow$$ the θ component cancels!

And the r component is simply

$$\vec{E} = Q\vec{e}_r\frac{1}{r^2}\left\{\frac{1}{\sqrt{1-\beta^2\sin^2\theta}} - \frac{\beta^2\cos^2\theta}{(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}\right\} = \frac{Q\vec{e}_r}{r^2}\left(\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}\right)$$

The $\vec{B}$ field similarly cancels in $\vec{e}_r$, and is $\vec{B} = \vec{\beta}\times\vec{E}$.