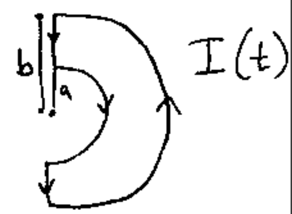
LECTURE 26

Retarded potentials: H&M square bracket notation:

$$[\rho(\vec{r}')] \equiv \rho(\vec{r}', t_{ret})$$ where you see "unnecessary" square brackets.

Remember that the actual potential is $\Phi(r,t), \vec{A}(\vec{r},t)$ – ie. these are the variables that you must take curls, gradients, etc. to find the fields. Note also that t_{ret} is not an independent variable.

A new example: A C-shaped wire.



What is $\vec{A}$ at origin?

$\Phi = 0$ since $\rho = 0$.

$$\vec{A} = \frac{1}{c}\left\{ \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \underbrace{r'}_{=a} d\phi' \frac{I(t-\frac{a}{c})}{a}\vec{e}_{\phi'} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{r'}_{=b} d\phi \frac{I(t-\frac{b}{c})}{b}\vec{e}_{\phi'} + 2\int_b^a dy' \frac{I(t-\frac{y'}{c})}{y'}\vec{e}_{y'} \right\}$$

$$= \frac{1}{c}\left\{ \left(I(t-\tfrac{a}{c}) - I(t-\tfrac{b}{c})\right) \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} d\phi' \vec{e}_{\phi'} + 2\int_b^a dy' \frac{I(t-\frac{y'}{c})}{y'}\vec{e}_{y'} \right\}$$

Can't integrate $\vec{e}_{\phi'} \Rightarrow$ but $\vec{e}_{\phi'} = \cos\phi' \vec{e}_y - \sin\phi' \vec{e}_x$ so $\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} d\phi' \vec{e}_{\phi'} = 2$

$$\Rightarrow \vec{A}(0) = \frac{2\vec{e}_y}{c}\left\{ I(t-\tfrac{a}{c}) - I(t-\tfrac{b}{c}) + \int_b^a dy' \frac{I(t-\frac{y'}{c})}{y'} \right\}$$

Special currents: $I = I_0 \cos\frac{2\pi c t}{b-a}$: $I(t-\frac{a}{c}) = I(t-\frac{b}{c})$ so $\vec{A}(0,t)$ determined entirely by the straight wires.

$I = gt$ with g constant: Integral separates into

$$\vec{A}(0) = \frac{2\vec{e}_y}{c}\left\{ \cancel{g\frac{b-a}{c}} + g\int_b^a dy' \left(\frac{t}{y'}\right) + \cancel{g\int_b^a dy' \left(-\frac{1}{c}\right)} \right\} = \boxed{\frac{2gt}{c}\ln\frac{b}{a}}$$

"Jefimenko" fields: Fields in terms of retarded sources:

$$\vec{E} = -\mathrm{grad}\,\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

What is $\frac{\partial \vec{A}}{\partial t}$? $\quad \vec{A} = \frac{1}{c}\int d^3r\, \frac{\vec{J}(\vec{r}, t_{ret})}{\mathscr{r}}$ ⟵ time dependent!

⟵ No time dependence

$$\frac{\partial}{\partial t}\left(J(\vec{r}, t_{ret})\right) = \frac{\partial \vec{J}}{\partial t_{ret}} \cdot \underbrace{\frac{\partial t_{ret}}{\partial t}}_{1}$$

so don't have to distinguish between $\frac{\partial [J]}{\partial t}$ and $\left[\frac{\partial J}{\partial t}\right]$.

Note that this only applies to partial derivative, since t_{ret} depends on $\vec{r}, \vec{r}\,'$ implicitly.

What is grad Φ ?

$$\mathrm{grad}\left\{\int d^3r'\, \frac{\rho(\vec{r}\,', t_{ret})}{\mathscr{r}}\right\}$$

grad commutes with $\int d^3r'$
(but grad' wouldn't):

$$= \int d^3r'\ \mathrm{grad}\left(\frac{\rho(\vec{r}\,', t_{ret})}{\mathscr{r}}\right) = \int d^3r' \left\{ \frac{1}{\mathscr{r}} \underbrace{\mathrm{grad}\,\rho(\vec{r}\,', t_{ret})}_{\substack{\text{NOT zero since} \\ t_{ret} \text{ depends on } \vec{r}\,!}} + \rho(\vec{r}\,', t_{ret})\,\mathrm{grad}\frac{1}{\mathscr{r}} \right\}$$

Recall $\mathrm{grad}\frac{1}{\mathscr{r}} = \frac{-\vec{\mathscr{r}}}{\mathscr{r}} = \frac{-\vec{e}_{\mathscr{r}}}{\mathscr{r}^2}$

By chain rule, $\mathrm{grad}\,\rho(\vec{r}\,', t_{ret}) = \underbrace{\mathrm{grad}\left(\rho(\vec{r}\,', t)\right)}_{0}\frac{\partial t_{ret}}{\partial t} + \frac{\partial \rho(\vec{r}\,', t_{ret})}{\partial t}\,\mathrm{grad}\,t_{ret}$

$$\Rightarrow \vec{E}(\vec{r},t) = \int d^3r' \left\{ \frac{\rho(\vec{r}\,',t_{ret})}{\mathcal{R}^2}\vec{e}_{\mathcal{R}} + \frac{1}{c\mathcal{R}}\frac{\partial \rho(\vec{r}\,',t_{ret})}{\partial t}\vec{e}_{\mathcal{R}} - \frac{1}{c^2\mathcal{R}}\frac{\partial \vec{J}(\vec{r}\,',t_{ret})}{\partial t} \right\}$$

By similar gymnastics.

$$\vec{B}(\vec{r},t) = \int d^3r' \left\{ \frac{1}{c\mathcal{R}^2}\vec{J}(\vec{r}\,',t_{ret})\times\vec{e}_{\mathcal{R}} + \frac{1}{c^2\mathcal{R}}\frac{\partial \vec{J}(\vec{r}\,',t_{ret})}{\partial t}\times\vec{e}_{\mathcal{R}} \right\}$$

Things to note: in static limit, (time derivatives all vanish) the non-Coulomb/Biot-Savart terms drop out.

In quasistatic limit, obtain Coulomb-Faraday law for $\vec{E}$:

$$\vec{E} = \int d^3r' \left(\frac{\rho}{\mathcal{R}^2}\vec{e}_{\mathcal{R}} - \frac{1}{c^2\mathcal{R}}\frac{\partial \vec{J}}{\partial t} \right)$$

with $t_{ret} \to t$.