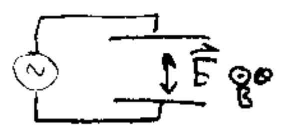
P3320 LECTURE 25

Retarded potentials: Recall the expressions for vector & scalar potentials from an arbitrary static distribution of charge & current:

$$\Phi(\vec{r}) = \int d^3r' \, \frac{\rho(\vec{r}')}{\mathscr{r}} \qquad \text{where } \vec{\mathscr{r}} \equiv \vec{r} - \vec{r}'$$

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{J}(\vec{r}')}{\mathscr{r}}$$

For moving or accelerated charges (which are associated with radiation), these potentials can't be sufficient. Example:



AC current into a capacitor.

$\vec{E}, \vec{B}$ fields locally are complicated — but far from the source we know they look at least partially like radiation, and we know that the fields then travel at the group velocity (c in free space). $\Rightarrow$ distant fields must depend on the charge configuation at an earlier time.

Invoke a heuristic argument to guess the potentials and fields at a distant point: introduce retarded time $t_{ret} \equiv t - \frac{\mathscr{r}}{c}$. This is the latest time a charge at $\vec{r}'$ could affect the fields at $\vec{r}, t$ if the field information travels at c. Now, we could write retarded versions of potentials, Coulomb, and Biot-Savart laws by replacing $\rho(\vec{r}', t)$ with $\rho(\vec{r}', t_{ret})$, as:

$$\Phi(\vec{r}, t) = \int d^3r \, \frac{\rho(\vec{r}', t_{ret})}{\mathscr{r}} \qquad \text{etc.}$$

If we do this, we will find an inconsistency:
curl $\vec{A}_{ret} \neq \vec{B}_{ret}$. So either $\vec{A}_{ret}$ is the correct potential, or $\vec{B}_{ret}$ is the right field (or neither!), but not both.
It can be shown using the wave equations in the Lorentz gauge (p.142 in your book, lecture (6)):

$$\nabla^2\begin{pmatrix}\Phi\\ \vec{A}\end{pmatrix} - \frac{1}{c}\frac{\partial^2}{\partial t^2}\begin{pmatrix}\Phi\\ \vec{A}\end{pmatrix} = -4\pi\begin{pmatrix}\rho\\ \vec{J}/c\end{pmatrix}$$

that the retarded potentials are correct. Nice proofs in both H&M and Griffiths (and they are rather different).

An example: the potential of a long wire, with current $I(t)$, along z axis:

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int d^3r' \, \frac{\vec{J}(\vec{r}',t_{ret})}{\mathscr{r}}$$

$$= \frac{1}{c}\int dz' \, \frac{I(t_{ret})\vec{e}_z}{\mathscr{r}}$$

For a current with $I(t<0)=0$, the only contribution is from regions where $t_{ret}>0$, or:

$$t_{ret} \equiv t - \frac{\sqrt{(z-z')^2+(r-r')^2}}{c}$$ but $r'=0$ since wire on axis.

$$\Longrightarrow ct > \sqrt{(z-z')+r^2}$$ take $z=0$: OK because of symmetry

$$(ct)^2 > z'^2 + r^2$$

Simplest case: $I = I_0\,\Theta(t)$ where $\Theta(x) \equiv 0 \quad x<0$
$\equiv 1 \quad x>0$

$$\vec{A}(\vec{r},t) = \frac{1}{c} I_0 \vec{e}_z \int_{-\sqrt{(ct)^2-r^2}}^{\sqrt{(ct)^2-r^2}} \frac{dz'}{\sqrt{r^2+z'^2}}$$

$$= \frac{2 I_0}{c} \vec{e}_z \ln\left(\frac{\sqrt{r^2+z'^2}+z'}{r}\right)\Bigg|_0^{\sqrt{(ct)^2-r^2}}$$

$$= \frac{2 I_0}{c} \vec{e}_z \ln \frac{ct+\sqrt{(ct)^2-r^2}}{r}$$

$$\Rightarrow \vec{E} = \frac{-\partial \vec{A}}{\partial t} = \frac{-2 I_0}{\sqrt{(ct)^2-r^2}} \vec{e}_z$$

$$\vec{B} = \text{curl}\,\vec{A} = -\frac{\partial A_z}{\partial r} \vec{e}_\phi = \frac{2 I_0}{\cancel{c} r} \frac{\cancel{c}t}{\sqrt{(ct)^2-r^2}} \vec{e}_\phi$$

$\longrightarrow$ At $t \to \infty$, recover $E = 0$, $\vec{B} = \frac{2 I_0}{cr} \vec{e}_\phi$

which are the static fields.