P3320 LECTURE 24

For a hollow waveguide, found Helmholtz equation for B_{0z}:

$$\left(\nabla_t^2 + k_c^2\right) B_{0z} = 0$$

... and the TE boundary condition $\left.\frac{\partial B_{0z}}{\partial n}\right|_S = 0$

(∂n is normal to surface S).

Rectangular waveguide: 

boundaries at $x = 0, a$; $y = 0, b$.

So, the conditions $\Rightarrow$ $x = 0, a: \frac{\partial B_{0z}}{\partial x} = 0$; $y = 0, b: \frac{\partial B_{0z}}{\partial y} = 0$.

$\Rightarrow$ rectangular sep. of variables problem. Solutions will have sin, cos dependence. Since deriv. $= 0$ at $x = 0, y = 0$, can set sines to zero:

$$B_{0z} = \sum_i \cos\alpha_i x \cos\beta_i y$$

In order to satisfy condition at $x = a, y = b$, need to set coefficients:

$$B_{0z} = \sum_{mn} B_{0mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{a}$$

m, n are mode numbers: One or the other can be zero, but not both (or the fields in x, y would vanish too). From the wave equation,

$$k_c^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \text{ ; wave mode called } TE_{mn}.$$

If $a > b$, lowest mode is TE_{10}, and $k_c = \frac{\pi}{a}$ $\Rightarrow$ largest allowed wavelength. Cutoff frequency is $\omega_c = ck_c = c\frac{\pi}{a}$

Generally, "clean" wave transport happens for frequencies where only TE_{10} mode is allowed:

$$\omega_{10} < \omega < \omega_{11} \quad \text{and} \quad \omega < \omega_{20}.$$

Transverse fields are given by derivative relations:

$$E_{0x} = \frac{ik_0}{k_c^2} \cancel{\frac{\partial B_{0z}}{\partial y}} \quad E_{0y} = \frac{-ik_0}{k_c^2}\frac{\partial B_{0z}}{\partial x} = \frac{-ik_0 B_0}{k_c^2}\frac{\partial}{\partial x}\cos\frac{\pi x}{a} = \frac{i\pi k_0 B_0}{a k_c^2}\sin\frac{\pi x}{a}$$

0 for TE_{10}

where $k_c = \frac{\pi}{a}$, so $E_{0y} = \frac{iak_0 B_0}{\pi}\sin\frac{\pi x}{a}$

Similarly $B_{0x} = \frac{ik_g}{k_c^2}\frac{\partial B_{0z}}{\partial x} = -ik_g\frac{aB_0}{\pi}\sin\left(\frac{\pi x}{a}\right)$

So: $\left|\frac{B_t}{E_t}\right| = \frac{k_g}{k_0}$ and $\vec{E}$ (in $\vec{e}_y$) remains normal to $\vec{B}$ (in $\vec{e}_x, \vec{e}_z$).

What is $\vec{B}$?

$$\vec{B} = e^{i(k_g z - \omega t)} B_0 \left(\cos\frac{\pi x}{a}\,\vec{e}_z - ik_g\frac{a}{\pi}\sin\frac{\pi x}{a}\,\vec{e}_x\right)$$

Note - $\frac{\pi}{2}$ out of phase! So cos and sin terms alternate — when sin term is imag., then $\vec{B}$ in z direction. When cos imag., then $\vec{B}$ in x direction. $e^{-i\omega t}$ factor keeps this changing at a given z.

Now, $k_g^2 = k_0^2 - k_c^2 \Longrightarrow k_g^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2$ so phase velocity

$$u_\phi = \frac{\omega}{k_g} = \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2}} = \frac{c}{\sqrt{1 - \frac{c^2\pi^2}{\omega^2 a^2}}} = \frac{c}{\sqrt{1 - \frac{\omega_{10}}{\omega^2}}} > c$$

But the group velocity $\frac{d\omega}{dk_g} = c\sqrt{1-\left(\frac{\omega_{10}}{\omega}\right)^2}$

$\Longrightarrow$ approaching cutoff frequency, group velocity approaches zero.

Circular waveguides: Helmholtz equation $\left(\nabla_t^2 + k_c^2\right)B_{0z} = 0$ again, but boundary condition is now $\left.\frac{\partial B_{0z}}{\partial r}\right|_S = 0$.

So, in cylindrical coordinates. $\nabla_t^2 \Leftrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$

so now we have $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} = -k_c^2 B_{0z}$

$\Longrightarrow$ Bessel's equation: Solutions are $\left(J_m(k_c r) \text{ and } N_m(k_c r)\right)\cos m\phi$

Where again, m is the mode #.

$\longrightarrow$ Can't have mode 0, since that would be TEM.