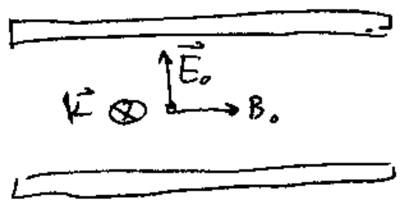
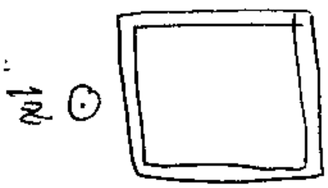
P3320 LECTURE 23

Hallow waveguides: First, revisit the TEM argument.



two planes: TEM allowed if $\vec{E}\perp$ and $\vec{B}\parallel$ to walls.

Now:



Clearly, fields have to be non-TEM if $\vec{E}, \vec{B}$ have no transverse dependence – but if their transverse dependence is constructed to have TEM and obey boundary cond., then you will find the fields have nonzero divergence.

(Nice argument in Griffiths too.)

For general solution, go to wave equations:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0 \text{ , and}$$

Now, restrict to solutions that propagate in z:

$$\vec{E} = \vec{E}_0(x,y)\, e^{i(k_g z - \omega t)}. \quad \text{where } k_g \text{ is group velocity}$$

$\hookrightarrow$ Now allow x, y dependence.

$$\Longrightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k_g^2 + \frac{\omega^2}{c^2}\right)\vec{E} = 0$$

Define a 2-D Laplacian: $\nabla_t^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

and perform separation of variables: use k_c^2 as sep. constant, where $k_0^2 = k_c^2 + k_g^2 = \frac{\omega^2}{c^2}$

$$-k_g^2 + \frac{\omega^2}{c^2} = -k_g^2 + k_0^2 = k_c^2$$

$$\Longrightarrow \left(\nabla_t^2 + k_c^2\right)\vec{E}_0 = 0.$$

Now, wave equations don't have all the information so need to enforce Maxwell's Equs too:

$$\mathrm{div}\,\vec{E} = \mathrm{div}\,\vec{B} = 0$$

$$\mathrm{curl}\,\vec{E} = \frac{-1}{c}\frac{\partial \vec{B}}{\partial t} = \frac{i\omega}{c}\vec{B} = ik_0\vec{B}$$

similarly, $\mathrm{curl}\,\vec{B} = -ik_0\vec{E}$

Now, remembering that $\vec{E}_0$ and $\vec{B}_0$ can have x, y dependence, we can solve for the fields and get the result (H&M Eq. 7.44-7.52):

$$E_{0x} = \frac{i}{k_c^2}\left(k_0\frac{\partial B_{0z}}{\partial y} + k_g\frac{\partial E_{0z}}{\partial x}\right) \quad E_{0y} = \frac{-i}{k_c^2}\left(k_0\frac{\partial B_{0z}}{\partial x} - k_g\frac{\partial E_{0z}}{\partial y}\right)$$

$$B_{0x} = \frac{-i}{k_c^2}\left(k_0\frac{\partial E_{0z}}{\partial y} - k_g\frac{\partial B_{0z}}{\partial x}\right) \quad B_{0y} = \frac{i}{k_c^2}\left(k_0\frac{\partial E_{0z}}{\partial x} + k_g\frac{\partial B_{0z}}{\partial y}\right)$$

$\longrightarrow$ Transverse components are a function of longitudinal components, from which it is obvious that TEM waves cannot propagate: $E_z = B_z = 0 \Longrightarrow \vec{E} = \vec{B} = 0$.

Guided TE waves: $E_z = 0$, $B_z \neq 0$:

$$E_{0x} = i\frac{k_0}{k_c^2}\frac{\partial B_{0z}}{\partial y}, \quad E_{0y} = \frac{-ik_0}{k_c^2}\frac{\partial B_{0z}}{\partial x}$$

$$\underbrace{B_{0x} = \frac{ik_g}{k_c^2}\frac{\partial B_{0z}}{\partial x}, \quad B_{0y} = \frac{ik_g}{k_c^2}\frac{\partial B_{0z}}{\partial y}}$$

These are the components of the transverse vector $\vec{B}_{t0}$

By inspection, can see $\vec{B}_{t0} = \frac{ik_g}{k_c^2}\,\mathrm{grad}\,B_{0z}$.

But we have another constraint from Maxwell curl equation:

$$\mathrm{curl}\,\vec{E} = ik_0\vec{B} \implies \underbrace{\cancel{\frac{\partial E_{0z}}{\partial y}}}_{0} - ik_gE_{0y} = ik_0B_{0x}$$

$$+ik_gE_{0x} - \underbrace{\cancel{\frac{\partial E_{0z}}{\partial x}}}_{0} = ik_0B_{0y}$$

$\longrightarrow$ Now we have also a pure transverse constraint:

$$\vec{B}_{t0} = \frac{k_g}{k_0}\left(\vec{e}_z \times \vec{E}_{t0}\right)$$

Similarly for TM waves,

$$\vec{E}_{t0} = \frac{-k_g}{k_0}\left(\vec{e}_z \times \vec{B}_{t0}\right) = \frac{ik_g}{k_c^2}\,\mathrm{grad}\,E_{0z}.$$

Now, we can find transverse fields knowing longitud., and find longitudinal fields by solving Helmholtz eq. and boundary cond:

$$\left(\nabla_t^2 + k_c^2\right)B_{0z} = 0, \quad \text{and } E_\parallel = B_\perp = 0 \text{ at boundary.}$$