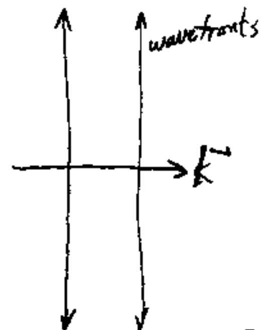
LECTURE 20

Guided waves: practical applications.

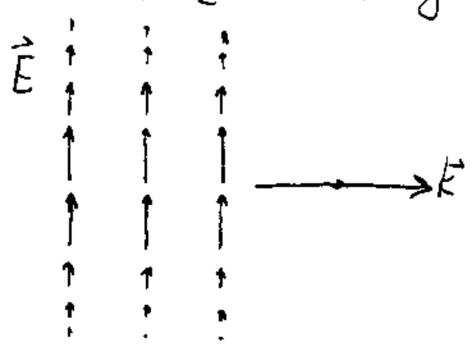
Often, want to use EM waves to transmit either power or signals over distances. Plane (or spherical) waves tend to be very inefficient: plane waves don't propagate forward unless they have infinite transverse extent: $\vec{E}(\vec{r}) = \vec{E}_0 \exp[i(\vec{k}\cdot\vec{r} - \omega t)]$ has



no dependence on coordinates transverse to $\vec{k}$. If we try to create a "beam" of plane waves with finite transverse extent, we have a problem:

$$\vec{E}(\vec{r}) = \vec{E}_0 \, e^{-\frac{x^2+y^2}{R^2}} \, e^{-i(kz-\omega t)}$$

looks like a wave with "plane" behavior in z but a gaussian-shaped falloff in x, y:



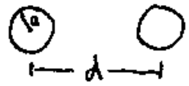
But you will find that this does not satisfy the wave equations. If you start a wave like this, it is really a superposition of waves of slightly varying $\hat{e}_k$, so over distance it will spread out and weaken: similar effect is diffraction.

Alternative: Use conductor boundary conditions to shape the transverse behavior of the field. Two major examples:

1. Transmission lines (twin-wire, coax, etc): 2 conductors
2. Hollow (usually) waveguides

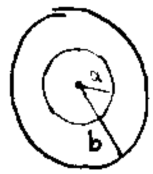
Two-conductor transmission lines: In general, a propagating wave confined by conductors will also involve currents in the conductors. In this case, it's customary to focus on the currents rather than the fields: Two very simple & important cases:

Twin-lead:



two parallel wires, radius a, distance d

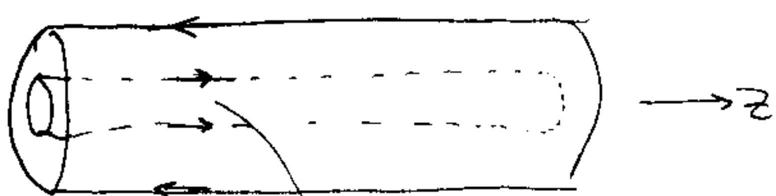
Coaxial:



two coaxial conductors.

Assuming the conductors perfect $(\sigma \to \infty)$, in the twin-lead case the wave field surrounds the wires, while in the coaxial case it is confined between the conductors.

Now, inject a pulse of + charge in one conductor, and equal − charge in the other:



charge density $\rho_\ell = +q(z,t)$

charge density $\rho_\ell = -q(z,t)$

From continuity equation, current & charge density related:

$$\frac{\partial \rho_\ell}{\partial t} = \frac{-\partial i(z,t)}{\partial z}$$

Note: i is <u>not</u> constant in z! Lower-case i for current: unfortunate notation.

More commonly, express this in terms of voltage rather than ρ_ℓ: $\rho_\ell = C_\ell V$ where C_ℓ = capacitance/length. Now:

$$(*) \qquad \frac{\partial V(z,t)}{\partial t} = -\frac{1}{C_\ell}\frac{\partial i(z,t)}{\partial z}.$$

But, note that i is not constant in time $\Rightarrow$ there is an inductive EMF too: $\mathcal{E} = -L\frac{\partial I}{\partial t}$.

Where i varies in z too, need to use $L_\ell = \frac{\text{inductance}}{\text{length}}$:

$$(**) \qquad \frac{\partial V(z,t)}{\partial z} = -L_\ell \frac{\partial i(z,t)}{\partial t}$$

Now we have coupled 1st-order PDEs for i, V. Take another derivative of each:

$$\frac{\partial^2 V}{\partial t^2} = \frac{-1}{C_\ell}\frac{\partial^2 i}{\partial z \partial t}, \quad \frac{\partial^2 V}{\partial z^2} = -L_\ell \frac{\partial^2 i}{\partial z \partial t}$$

$$\Longrightarrow \quad \frac{\partial^2 V}{\partial z^2} = L_\ell C_\ell \frac{\partial^2 V}{\partial t^2}$$

Similarly, $$\frac{\partial^2 i}{\partial z^2} = L_\ell C_\ell \frac{\partial^2 i}{\partial t^2}$$

So we have wave equations in V, i now, and the phase velocity is simply $u = \frac{1}{\sqrt{L_\ell C_\ell}}$ where L_ℓ and C_ℓ depend on the cross-sectional geometry of the line.

(note that H&M call the velocity c — but it's not c.)

Now, consider a periodic signal:

$$V(0,t) = V_0 e^{-i\omega t}$$

At $z>0$, $\quad V = V_0 e^{i(kz-\omega t)} \quad$ where $k = \frac{\omega}{u} = \omega\sqrt{L_\ell C_\ell}$

Use one of the coupled first-order PDEs to find $i(z,t)$:

$$\frac{\partial i}{\partial t} = -\frac{1}{L_\ell}\frac{\partial V}{\partial z} \quad \Rightarrow \quad \frac{\partial i}{\partial t} = -\frac{ik}{L_\ell} e^{i(kz-\omega t)}$$

$$\text{so} \quad i(z,t) = \sqrt{\frac{C_\ell}{L_\ell}}\, V_0 e^{i(kz-\omega t)}.$$

Define the impedance of the line as $Z_0 \equiv \frac{V(0,t)}{i(0,t)} = \sqrt{\frac{L_\ell}{C_\ell}}$.

If the impedance changes abruptly along the line, there will be a reflected wave.