P3320 LECTURE 14

Exam information:

- Thursday Oct. 4, 7:00 – 8:30 pm in G2B47
- Material: Chapters 1, 4, 5 except Sec. 4.9, 4.10
- Reviews: Monday lecture (Oct. 1) – topical review
  Monday PM session (Oct. 1) – Q&A 5–7 pm

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Reflection and refraction:

Start by considering boundary conditions on the field vectors at an interface between linear media, where there is no free charge or free current:

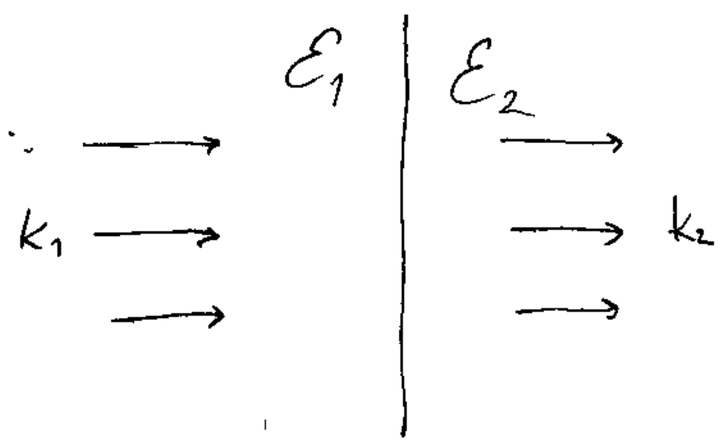
- Where divergence non-infinite, normal component of vector field is continuous at boundary
- Where curl non-infinite, transverse component continuous

So with no free current or charge, $\operatorname{div}\vec{D} = \operatorname{div}\vec{B} = 0$,
$\operatorname{curl}\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ and $\operatorname{curl}\vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$ neither of which can blow up

So at boundary of materials 1 and 2,

$$E_{1\parallel} = E_{2\parallel}, \quad D_{1\perp} = D_{2\perp}, \quad B_{1\perp} = B_{2\perp}, \quad H_{1\parallel} = H_{2\parallel}.$$

Now, consider plane wave propagating normal to the boundary between two perfect dielectrics: $\mu = 1$, $\sigma = 0$, $\varepsilon = \varepsilon_1$ or $\varepsilon_2$.
Assume the boundary is $z = 0$, and the "incident" wave originates at $z = -\infty$ and is propagating with wave number $\vec{k} = k\vec{e}_z$, and wave is linearly polarized in the $\vec{e}_x$ direction.




Since the incoming wave is monochromatic, only need to consider components of one frequency. Also by symmetry, only have waves propagating in $\pm z$ direction. So we know that:

→ $z<0$: have wave in $+z$ direction, plus potentially reflected wave in $-z$ direction with a phase shift. $\frac{\omega}{k_1} = c/\sqrt{\varepsilon_1}$ here.

→ $z>0$: have wave in $+z$ direction with $\frac{\omega}{k_2} = c/\sqrt{\varepsilon_2}$. No wave in $-z$ direction since no source at $z=+\infty$.

So we know: $\vec{E}_{left} = \vec{E}_{in} + \vec{E}_{refl}$. $\quad \vec{B}_{left} = \vec{B}_{in} + \vec{B}_{refl}$.

$\vec{E}_{right} = \vec{E}_{trans} \quad \vec{B}_{right} = \vec{B}_{trans}$

$\vec{E}_{in} = E_0^{in}\vec{e}_x e^{i(k_1 z - \omega t)}$

$\vec{E}_{refl} = E_0^{refl}\vec{e}_x e^{i(-k_1 z - \omega t)} e^{i\delta_{refl}}$ — Absorb these phases into

$\vec{E}_{trans} = E_0^{trans}\vec{e}_x e^{i(k_2 z - \omega t)} e^{i\delta_{trans}}$ — the coefficients: $E_{refl}e^{i\delta_{refl}} \to E_{refl}$,

$\vec{H}_{in} = H_{in}\vec{e}_y \cdots$ etc, with $|H_{in}| = \sqrt{\varepsilon}|E_{in}|$. which is understood complex.

Now, constrain things with the boundary conditions:

$\vec{E}_{z<0} = \vec{e}_x e^{-i\omega t}\left(E_0^{in}e^{ik_1 z} + E_0^{refl}e^{-ikz}\right)$, $\quad \vec{E}_{z>0} = \vec{e}_x e^{-i\omega t}\left(E_0^{trans}e^{ikz}\right)$

At $z=0$, $E_x^{trans} = E_x^{in} + E_x^{refl}$ :

$$e^{-i\omega t}\left(E_0^{in} + E_0^{refl}\right) = e^{-i\omega t}\left(E_0^{trans}\right)$$

$$\Rightarrow E_0^{in} + E_0^{refl} = E_0^{trans}$$

$H_0^{refl} = -n E_0^{refl}$ so that $\vec{E}\times\vec{H}$ is in $-z$ direction.

Same for H: $H_0^{in} + H_0^{refl} = H_0^{trans} \Rightarrow \sqrt{\varepsilon_1}\left(E_0^{in} - E_0^{refl}\right) = \sqrt{\varepsilon_2}\left(E_0^{trans}\right)$

$\sqrt{\varepsilon_1} = n_1$, $\sqrt{\varepsilon_2} = n_2$ :

$$E_0^{in} + E_0^{refl} = E_0^{trans}$$
$$n_1\left(E_0^{in} - E_0^{refl}\right) = n_2 E_0^{trans}$$

Solving simultaneously: $n_1\left(E_0^{in} - E_0^{refl}\right) = n_2\left(E_0^{in} + E_0^{refl}\right)$

$$(n_1 - n_2)E_0^{in} = (n_1 + n_2)E_0^{refl}.$$

or $$E_0^{refl} = \frac{n_1 - n_2}{n_1 + n_2}E_0^{in}$$

$\Rightarrow$ $\vec{E}^{refl}$ has a phase shift of $\pi$ if $n_2 > n_1$
$\vec{B}^{refl}$ has a phase shift of $\pi$ if $n_1 > n_2$.

Total intensity $\frac{\text{reflected}}{\text{incident}}$ is $\frac{\langle |E\times H|^{refl}\rangle}{\langle |E\times H|^{in}\rangle} = \frac{|E_0^{refl}|^2}{|E_0^{in}|^2} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2.$

call this $R$ = reflection coefficient

Transmitted fraction $= \frac{\langle |E\times H|^{trans}\rangle}{\langle |E\times H|^{in}\rangle} = \frac{n_2 |E_0^{trans}|^2}{n_1 |E_0^{in}|^2} \equiv T$

**Solving**: $E_0^{trans} = E_0^{in} + E_0^{refl} = E_0^{in} + \left(\frac{n_1 - n_2}{n_1 + n_2}\right)E_{in} = E_{in}\left(\frac{2n_1}{n_1 + n_2}\right)$

so $T = \frac{n_2}{n_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2 = \frac{4n_1 n_2}{(n_2 + n_1)^2}$

We had better have energy conservation: $T+R=1$. Check:

$$\frac{4n_1n_2}{(n_2+n_1)^2}+\left(\frac{n_1-n_2}{n_1+n_2}\right)^2 = \frac{4n_1n_2+n_1^2+n_2^2-2n_1n_2}{(n_2+n_1)^2} = \frac{(n_1^2+n_2^2)}{(n_1^2+n_2^2)} \checkmark$$

Note that $T$ and $R$ do not depend on <u>which</u> medium has higher index of refraction! But the phase shift of $\vec{E}$ vs. $\vec{B}$ is different:

Reflection off <u>more</u> refractive material: sign of $\vec{E}$ reverses.

Reflection off <u>less</u> refractive material: sign of $\vec{B}$ reverses.