P3320 LECTURE 12 + 13

How does an EM wave propagate in a conductor? This material (potentially) has ε-1, μ-1, σ all nonzero. We will still assume no free static charge, but now:

$$\vec{J}_f = \sigma\vec{E} \neq 0.$$

Ampere-Maxwell law now becomes:

$$\text{curl}\,\vec{H} = \frac{\varepsilon}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\left(\sigma\vec{E}\right)$$

so
$$\text{curl}\,\vec{B} = \frac{\varepsilon\mu}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi\sigma\mu}{c}\vec{E}$$

since we are assuming linear, homogeneous materials.

Can now propagate this change through to the wave equations:

$$\nabla^2\vec{E} = \frac{\varepsilon\mu}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \frac{4\pi\sigma\mu}{c^2}\frac{\partial\vec{E}}{\partial t}$$

$$\nabla^2\vec{B} = \frac{\varepsilon\mu}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} + \frac{4\pi\sigma\mu}{c^2}\frac{\partial\vec{B}}{\partial t}$$

Things to note here:

- $\vec{E}$ and $\vec{B}$ (second-order) equations are still uncoupled
- $\vec{E}$ and $\vec{B}$ equations still have identical form.

Remember we used exponential form to solve vacuum wave eqn because it is proportional to its second derivative. Fortunately it's also proportional to its first derivative. So take $\vec{E} = \vec{E}_0 \exp[i(k\xi - \omega t)]$ and plug it in (using H&M notation where ξ is the $\hat{e}_k$ coordinate):

$$\frac{\partial^2 \vec{E}}{\partial \xi^2} = (ik)^2 \vec{E} = \frac{\varepsilon\mu}{c^2}(-i\omega)^2 \vec{E} + \frac{4\pi\sigma\mu}{c^2}(-i\omega)\vec{E}$$

$$-k^2 = -\frac{\varepsilon\mu}{c^2}\omega^2 - i\frac{4\pi\sigma\mu}{c^2}\omega \,.$$

or $k^2 = \frac{\varepsilon\mu\omega^2}{c^2}\left(1 + \frac{4\pi\sigma}{\varepsilon\omega} i\right)$ So k is complex.

Note a few things about this result:

- Write $k \equiv \alpha + i\beta$. Now, $e^{i(k\xi - \omega t)}$ has following behavior: $\vec{E} = \vec{E}_0 e^{-\beta\xi} e^{i(\alpha\xi - \omega t)}$

  Wave is attenuated by $e^{-\beta\xi}$ as it travels through the material: amplitude drops by factor of e every $\frac{1}{\beta}$.
  (This is called <u>skin depth</u> $\delta \equiv 1/\beta$.)
- Taking the square root (see textbook for details):

$$\alpha = \frac{\omega}{c}\left[\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \left(\frac{4\pi\sigma}{\mu\varepsilon}\right)^2} + 1\right)\right]^{\frac{1}{2}}$$

$$\beta = \frac{\omega}{c}\left[\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \left(\frac{4\pi\sigma}{\mu\varepsilon}\right)} - 1\right)\right]^{\frac{1}{2}}$$

k is no longer proportional to ω: plane waves of different frequencies propagate with different speeds. (dispersion)

So we have to be clear that this formulation only works for monochromatic waves with a particular ω, unlike the previous discussion of waves in nonconductors, where $\frac{\omega}{k}$ was a constant. Here, we must handle any wave as a superposition of monochromatic waves.

In this framework, we can use a trick to hide some of the complication:

$$\text{curl}\vec{B} = \frac{\varepsilon\mu}{c}\frac{\partial\vec{E}}{\partial t} \quad \text{in nonconductors.} \quad \text{If } \vec{E} = \vec{E}_0 e^{-i(\vec{k}\cdot\vec{r}-\omega t)}$$

then $\text{curl}\vec{B} = \frac{\varepsilon\mu}{c}(-i\omega)\vec{E}$.

In conductor case, we have then $\text{curl}\vec{B} = \frac{\varepsilon\mu}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi\sigma\mu}{c}\vec{E}$,

which looks like above with $\frac{\varepsilon\mu}{c}(-i\omega)$ replaced with something complex:

$$\text{curl}\vec{B} = \left[\frac{\varepsilon\mu}{c}(-i\omega) + \frac{4\pi\sigma\mu}{c}\right]\vec{E}$$

so define complex dielectric "constant" $\hat{\varepsilon} \equiv \varepsilon + i\frac{4\pi\sigma}{\omega}$ which is dependent on ω! Now, for waves of frequency ω only,

$$\text{curl}\vec{B} = \frac{\hat{\varepsilon}\mu}{c}\frac{\partial\vec{E}}{\partial t} \quad \text{in a conductor too.}$$

Now, (complex) $\hat{k} = \frac{\omega}{c}\sqrt{\hat{\varepsilon}\mu}$

and (complex) $\hat{n} \equiv \sqrt{\hat{\varepsilon}\mu}$ complex index of refraction

Note that imag. part of $\hat{\varepsilon}$ is responsible for absorption of wave

Can use $\hat{n}$ to see effect on fields of conductor:

$$\vec{B} = n\vec{e}_k \times \vec{E}$$
$$\vec{E} = -\frac{1}{n}\vec{e}_k \times \vec{B}$$
yielded $\frac{|\vec{B}_0|}{|\vec{E}_0|} = n$ for nonconductors.

Taking $n \to \hat{n}$, and writing $\hat{n} = |\hat{n}|e^{i\lambda}$, where $\lambda = \arctan\frac{\beta}{\alpha}$

$$\vec{B} \to \left(|\hat{n}|\,\vec{e}_k \times \vec{E}_0\right)e^{i\lambda}$$
(note this is $\frac{1}{2}$ the phase of $\hat{\varepsilon}$)

$$\vec{E} \to \left(\frac{-1}{|\hat{n}|}\vec{e}_k \times \vec{B}_0\right)e^{-i\lambda}$$

So $\vec{B},\vec{E}$ are no longer in phase, and $\frac{|\vec{B}_0|}{|\vec{E}_0|} = |\hat{n}| = \sqrt{\mu}\left(\varepsilon^2 + \frac{16\pi^2\sigma^2}{\omega^2}\right)^{1/4}$

Limiting cases: $\hat{k}^2 = \frac{\varepsilon\mu\omega^2}{c^2}\left(1 + i\,\frac{4\pi\sigma}{\varepsilon\omega}\right)$

1) Low σ or high frequency: dispersive term dominates $\left(\frac{4\pi\sigma}{\varepsilon\omega} \ll 1\right)$:

Can use series expansion of $(1+x)^{\frac{1}{2}}$ to find the complex k to first order in σ: $k = \frac{\omega\sqrt{\varepsilon\mu}}{c}\left(1 + i\frac{2\pi\sigma}{\varepsilon\omega} - \dots\right)$

Now $\mathrm{Im}(k) = \frac{1}{\delta} = \frac{2\pi\sigma}{c}\sqrt{\frac{\mu}{\varepsilon}}$ which is independent of ω.

2) High conductivity: absorptive term dominates $\left(\frac{4\pi\sigma}{\varepsilon\omega} \gg 1\right)$. Typical metals are in this category. $k^2 \cong \frac{4\pi i\sigma\mu\omega}{c^2}$

So $k \approx \frac{\sqrt{4\pi\sigma\mu\omega}}{c}\sqrt{i}$ where $\sqrt{i} = \sqrt{e^{i\pi/2}} = e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$$= \frac{\sqrt{2\pi\sigma\mu\omega}}{c}(1+i)$$

So absorptive and dispersive terms have same magnitude — absorptive isn't dominant as might naively guess.

But — notice that phase velocity depends heavily on σ:

$$\frac{\omega}{\mathrm{Re}(k)} = \frac{c\sqrt{\omega}}{\sqrt{2\pi\sigma\mu}}$$ rather than $\frac{c}{n}$: phase velocity is reduced by a factor of $\sqrt{\frac{2\pi\sigma}{\omega}}$.

But skin depth is $\frac{1}{\mathrm{Im}(\hat{k})} = \frac{c}{\sqrt{2\pi\sigma\omega\mu}}$ too, so both wavelength and skin depth scale by the same factor: $\lambda = \frac{2\pi}{\mathrm{Re}(k)}$ so $\delta = \frac{\lambda}{2\pi}$. Since $|\hat{n}| \gg 1$, $\frac{|\vec{B}_0|}{|\vec{E}_0|} \gg 1$ too: $\vec{E}$ is small.

Example: radio waves in non-ferromagnetic metals. Typical conductivity is $5\times10^{17}\ s^{-1}$. Assume $\mu = 1$.

Typical AM radio waves are 1 MHz, so $\omega = 2\pi\cdot10^{6}\ s^{-1}$.

In vacuum, then $c = \frac{\omega}{k} \Rightarrow k \cong 2\times10^{-4}\ \mathrm{cm}^{-1}$, and

$\lambda = \frac{2\pi}{k} \cong 3\times10^{4}\ \mathrm{cm} = 300\ \mathrm{m}$.

In metal, $$\frac{1}{\mathrm{Re}(\hat{k})} \approx \frac{1}{\mathrm{Im}(\hat{k})} = \frac{c}{\sqrt{2\pi\sigma\omega}} = \frac{3\cdot10^{10}\ \mathrm{cm/s}}{\sqrt{2\pi\cdot5\cdot10^{17}s^{-1}\cdot2\pi\cdot10^{6}s^{-1}}}$$

so $\delta = 0.007\ \mathrm{cm}$!

Also, λ becomes very small (not that it's a very relevant concept when amplitude is attenuated by $e^{-2\pi} \approx \frac{1}{500}$ in each λ.)

So, AM radio is easily blocked by a few sheets of aluminum foil.

Skin effect also applies to driving an alternating current through a conductor. In a good conductor, wave equation for $\vec{E}$ is

$$\nabla^2 \vec{E} \approx \frac{4\pi\sigma\mu}{c^2}\frac{\partial \vec{E}}{\partial t}$$

since the $\frac{\partial^2 E}{\partial t^2}$ is small without a σ in the numerator.

Taking $\vec{J} = \sigma\vec{E}$, this becomes a diffusion equation for $\vec{J}$:

$$\nabla^2 \vec{J} = \frac{4\pi\sigma\mu}{c^2}\frac{\partial \vec{J}}{\partial t}.$$

For "steady alternating current," $\vec{J}(\vec{r},t) = \vec{J}_0(\vec{r})\,e^{-i\omega t}$

So $\frac{\partial \vec{J}}{\partial t} = -i\omega\vec{J}$

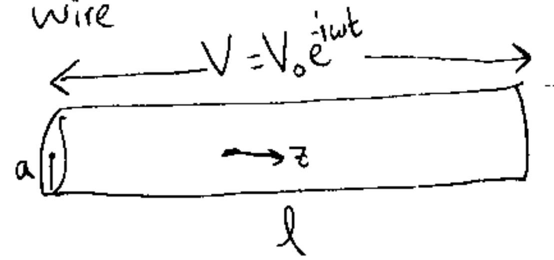
and $\nabla^2\vec{J} = -\frac{4\pi\sigma\mu\omega}{c^2} i\vec{J}$. Mult. both sides by $e^{i\omega t}$:

$$\nabla^2 \vec{J}_0 = \underbrace{-\frac{4\pi\sigma\mu\omega}{c^2} i}_{\text{H\&M call this } \tau^2} \vec{J}_0$$

Since τ^2 is pure imaginary, use same $\sqrt{i}$ formula to find

$$\tau = (1+i)\frac{\sqrt{2\pi\sigma\mu\omega}}{c} = \frac{1+i}{\delta}.$$

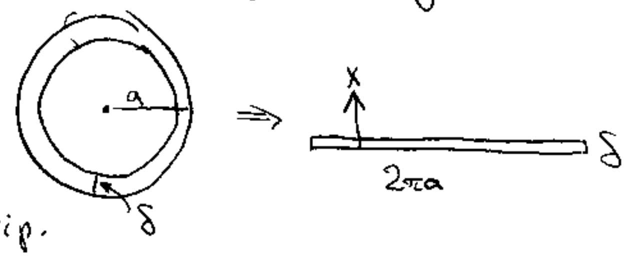
Treated exactly in H&M, with Bessel functions: here, calculate in the small-δ (high-frequency) limit: current distribution in a wire



Assume $\delta \ll a$.

$\vec{E}$ at surface is $\vec{E} = E_0 e^{i\omega t} \vec{e}_z$, where $E_0 = \frac{V_0}{\ell}$.

$\vec{E}$ inside will drop as $e^{-(a-r)/\delta}$, and by assuming $\delta \ll a$ we can "unroll" the wire:



So we have a strip $2\pi a$ wide, and $\vec{E} = E_0 e^{-i\omega t} e^{-x/\delta}$ in the strip.

What is total I? Can assume x can go to ∞ since $J \to 0$ quickly.

$$\vec{J} = \sigma \vec{E} = \sigma E_0 e^{-i\omega t} e^{-x/\delta} \vec{e}_z$$

$$I = 2\pi a \int_0^\infty dx\, \sigma E_0 e^{-i\omega t} e^{-x/\delta} = 2\pi a \sigma E_0 e^{-i\omega t} (-\delta)(-1)$$

$$= 2\pi a \sigma \delta \left(\vec{E}_{outside} \cdot \vec{e}_z \right)$$

$$= 2\pi a \sigma \delta \frac{V}{\ell}$$

So, at high frequency, current only flows along the outside "skin" of a conductor. Interpret this as an effective resistance: $R_{AC} = \frac{V}{I} = \frac{\ell}{2\pi a \sigma \delta}$

Compared to DC resistance $R_{DC} = \frac{1}{\sigma} \frac{\ell}{\pi a^2}$

$$\frac{R_{AC}}{R_{DC}} = \frac{a}{2\delta}.$$