P3320 FINAL EXAM SOLUTIONS

(1) Start by transforming the radiation into the cube's rest frame (call this the primed frame):

$$\vec{E}_0 = E_0\vec{e}_x, \quad \vec{B}_0 = E_0\vec{e}_y$$

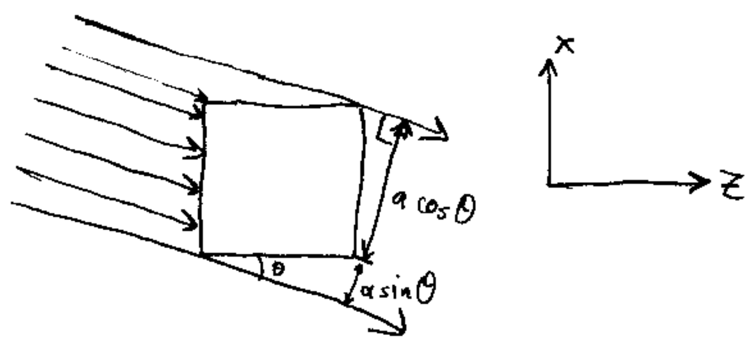
$$\text{So } \vec{E}_0' = E_0\vec{e}_x + \beta\gamma E_0\vec{e}_z, \quad \vec{B}_0' = \gamma E_0(\vec{e}_y).$$

$$\vec{E}_0' \times \vec{B}_0' = \gamma E_0^2(\vec{e}_x \times \vec{e}_y) + \beta\gamma^2 E_0^2(\vec{e}_z \times \vec{e}_y)$$

$$= E_0^2(\gamma\vec{e}_z - \beta\gamma^2\vec{e}_x)$$

This represents radiation with intensity scaled up by γ^2 (you can check this: $\sqrt{\gamma^2 + (\beta\gamma^2)^2} = \gamma^2$.) It is incident on the cube at an angle $\tan\Theta = \frac{\beta\gamma^2}{\gamma}$ from the z axis.

Now, the cube's rocket has to provide enough force to oppose the radiation pressure. Since the cube is fully absorbing, it absorbs the entire momentum of the radiation striking it. Therefore the force is $\frac{\gamma^2 E_0^2}{8\pi}A$ where A is the cross-sectional area of the cube normal to the radiation:



Since $\tan\Theta = \beta\gamma$, $\sin\Theta = \beta$ and $\cos\Theta = \frac{1}{\gamma}$. (Same angle as in prob. set!)

$$A = a^2\left(\beta + \frac{1}{\gamma}\right) \Rightarrow F = \frac{\gamma^2 E_0^2 a^2}{8\pi}\left(\beta + \frac{1}{\gamma}\right) \text{ in the } \pi - \theta \text{ direction from } \vec{e}_z.$$

2. Solution: for the coaxial cylinder we are to find the capacitance C and inductance L per length.

From Gauss' Law, denote the linear charge density by λ

$$\oint \vec{D}\cdot d\vec{A} = 4\pi Q$$

$$\Rightarrow D\cdot 2\pi r \cdot l = 4\pi\lambda l$$

$$\Rightarrow \vec{D} = \frac{2\lambda}{r}\vec{e}_r$$

So $\vec{E} = E(r)\vec{e}_r$ where $E(r) = \begin{cases} \frac{2\lambda}{\epsilon_1 r} & a<r<g \\ \frac{2\lambda}{\epsilon_2 r} & g<r<b \end{cases}$

$$\Phi = \int_a^b \vec{E}\cdot d\vec{l} = \frac{2\lambda}{\epsilon_1}\ln\frac{g}{a} + \frac{2\lambda}{\epsilon_2}\ln\frac{b}{g}$$

$$\therefore C = \frac{Q}{\Phi} = \frac{\lambda}{\Phi} = \frac{1}{\frac{2}{\epsilon_1}\ln\frac{g}{a}+\frac{2}{\epsilon_2}\ln\frac{b}{g}}$$ per length

Also it could be viewed as two capacitances connected in series: $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$ where $C_i = \frac{\epsilon_i}{2\ln\frac{r_>}{r_<}}$

Where $\begin{cases} r_> \\ r_< \end{cases}$ are $\begin{cases} \text{inner} \\ \text{outer} \end{cases}$ radii of the cylinder.

for L: since $\mu = 1$ for both media, We proceed as in the first Exam that $L = \frac{2}{c^2}\ln\frac{b}{a}$ per length.

So the impedance

$$Z=\sqrt{\frac{L}{C}}=\frac{2}{c}\sqrt{\ln\left(\frac{b}{a}\right)\left(\frac{1}{\varepsilon_1}\ln\frac{g}{a}+\frac{1}{\varepsilon_2}\ln\frac{b}{g}\right)}$$

and for group velocity i.e. the transmission velocity:

$$v_g=\sqrt{\frac{1}{LC}}=c\sqrt{\frac{\frac{1}{\varepsilon_1}\ln\frac{g}{a}+\frac{1}{\varepsilon_2}\ln\frac{b}{g}}{\ln(b/a)}}$$

(3) Initial four-momentum is $p^{\mu} = p_A^{\mu} + p_B^{\mu}$

(call the daughter particles A and B to reduce index confusion)

$$= \left(\frac{E_A + E_B}{c},\ p_A^1 + p_B^1,\ p_A^2 + p_B^2,\ p_A^3 + p_B^3 \right)$$

where $\frac{E_A}{c} = \sqrt{m_A^2 c^2 + p_A^2}$, $\frac{E_B}{c} = \sqrt{m_B^2 c^2 + p_B^2}$

Now, $-(Mc)^2 = p_{\mu} p^{\mu}$

$$(Mc)^2 = \left(\frac{E_A + E_B}{c}\right)^2 - \left(p_A^1 + p_B^1\right)^2 - \left(p_A^2\, p_B^2\right)^2 - \left(p_A^3\, p_B^3\right)^2$$

or $$M^2 = \frac{\left(\sqrt{m_A^2 c^2 + p_A^2} + \sqrt{m_B^2 c^2 + p_B^2}\right)^2}{c^2} - \frac{1}{c^2}\left(\vec{p}_A + \vec{p}_B\right)\cdot\left(\vec{p}_A + \vec{p}_B\right)$$

$$M^2 = \frac{m_A^2 c^2 + \cancel{p_A^2} + m_B^2 c^2 + \cancel{p_B^2} + 2\sqrt{(m_A^2 c^2 + p_A^2)(m_B^2 c^2 + p_B^2)}}{c^2} - \frac{1}{c^2}\left(\cancel{p_A^2} + \cancel{p_B^2} + 2\vec{p}_A \cdot \vec{p}_B\right)$$

$$M^2 = m_A^2 + m_B^2 + \frac{2\sqrt{(m_A^2 c^2 + p_A^2)(m_B^2 c^2 + p_B^2)}}{c^2} - \frac{2\vec{p}_A \cdot \vec{p}_B}{c^2}$$

④ ⓐ We know $\vec{p} = e\vec{d}$ where $\vec{d}$ is the separation vector of the charges. In this case, $\vec{d} = -a[\cos(\omega t)\vec{e}_x + \sin(\omega t)\vec{e}_y]$. So clearly $p_0 = -ea$. (or $+ea$ — the dipole is then defined with a phase shifted by π.)

ⓑ $$\ddot{\vec{p}} = p_0\left[-\omega^2\cos(\omega t)\vec{e}_x - \omega^2\sin(\omega t)\vec{e}_y\right] = -\omega^2\vec{p}.$$

Know $\vec{B}_{rad} = \frac{-1}{c^2 r}\vec{e}_r \times \ddot{\vec{p}}(t_{ret})$ (Lecture 33)

$$\vec{E}_{rad} = \vec{B}_{rad} \times \vec{e}_r = \frac{-1}{c^2 r}\left(\vec{e}_r \times \ddot{\vec{p}}(t_{ret})\right) \times \vec{e}_r = \frac{1}{c^2 r}\vec{e}_r \times \left(\vec{e}_r \times \ddot{\vec{p}}(t_{ret})\right)$$

$$= \frac{1}{c^2 r}\left[(\vec{e}_r \cdot \ddot{\vec{p}})\vec{e}_r - (\vec{e}_r \cdot \vec{e}_r)\ddot{\vec{p}}(t_{ret})\right]$$

$$= \frac{1}{c^2 r}\left[\left(\vec{e}_r \cdot \ddot{\vec{p}}(t_{ret})\right) - \ddot{\vec{p}}\right]$$ (This is also given as Griffiths Eq. 11.56)

$$\vec{E}_{rad} = \frac{-p_0\omega^2}{c^2 r}\left\{\cos\left[\omega\left(t - \frac{r}{c}\right)\right](\vec{e}_x \cdot \vec{e}_r)\vec{e}_r + \sin\left[\omega\left(t - \frac{r}{c}\right)\right](\vec{e}_y \cdot \vec{e}_r)\vec{e}_r - \vec{p}(t_{ret})\right\}$$

$$\vec{E}_{rad} = \frac{-p_0\omega^2}{c^2 r}\left\{\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\left(\frac{x}{r}\vec{e}_r - \vec{e}_x\right) + \sin\left[\omega\left(t - \frac{r}{c}\right)\right]\left(\frac{y}{r}\vec{e}_r - \vec{e}_y\right)\right\}$$

QED.

ⓒ Since $\vec{B}_{rad} = \vec{e}_r \times \vec{E}_{rad}$, $|E_{rad}| = |B_{rad}|$ and both normal to $\vec{e}_r$.

So $\langle S \rangle = \frac{c}{4\pi}\langle E^2 \rangle$ where $\langle\ \rangle$ denotes time average.

$$= \frac{c\vec{e}_r}{4\pi}\frac{\omega^4}{c^4 r^2}\left((\vec{p}\cdot\vec{e}_r)\vec{e}_r - \vec{p}\right)^2_{avg}$$

$$= \frac{\vec{e}_r\,\omega^4}{4\pi c^3 r^2}\left((\vec{p}\cdot\vec{e}_r)^2 + p^2 - 2\vec{p}\cdot(\vec{p}\cdot\vec{e}_r)\vec{e}_r\right)_{avg}$$

$$\langle S\rangle = \frac{\vec{e}_r p_0^2 \omega^4}{4\pi c^3 r^2}\left\{1-\sin^2\theta\left[\left(\cos\phi\,\vec{e}_x+\sin\phi\,\vec{e}_y\right)\cdot\left(\cos(\omega t_{ret})\vec{e}_x+\sin(\omega t_{ret})\vec{e}_y\right)\right]^2\right\}_{avg}$$

$$= \frac{\vec{e}_r p_0^2 \omega^4}{4\pi c^3 r^2}\left\{1-\sin^2\theta\left[\cos\phi\cos(\omega t_{ret})+\sin\phi\sin(\omega t_{ret})\right]^2\right\}_{avg}$$

$$= \frac{\vec{e}_r p_0^2 \omega^4}{4\pi c^3 r^2}\left[1-\sin^2\theta\underbrace{\cos^2(\omega t_{ret}+\phi)}_{\text{Averages to } \frac{1}{2}}\right]_{avg}$$

$$= \frac{\vec{e}_r p_0^2 \omega^4}{4\pi c^3 r^2}\left(1-\frac{1}{2}\sin^2\theta\right)$$

$$\langle P\rangle = \int d\Omega\left(4\pi r^2\langle S\rangle\right) = \frac{p_0^2\omega^4}{2c^3}\int_{-1}^{1} d\cos\theta\left(1-\frac{1}{2}(1-\cos^2\theta)\right)$$

$$= \frac{p_0^2\omega^4}{2c^3}\int_{-1}^{1} du\left(\frac{1}{2}+\frac{u^2}{2}\right) = \frac{p_0^2\omega^4}{2c^3}\left(\frac{8}{6}\right) = \boxed{\frac{2p_0^2\omega^4}{3c^3}}$$

...which is twice the Larmor power of a single oscillating dipole of strength p_0. Note that this is only true in this special case where the "two dipoles" are orthogonal and out of phase — normally there would be interference between their fields.