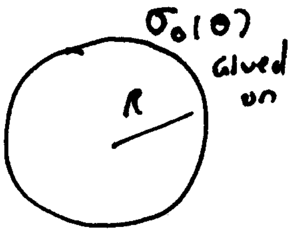
3310 - 3-32 b

One last example:



$$V_{out}(r,\theta) = \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$V_{in}(r,\theta) = \sum_l A_l r^l P_l(\cos\theta)$$

Continuity says $A_l R^l = \dfrac{B_l}{R^{l+1}}$ (each term matches!)

$$\left.\frac{\partial V}{\partial r}\right|_{out, R} - \left.\frac{\partial V}{\partial r}\right|_{in, R} = -\frac{\sigma_0}{\varepsilon_0} \Rightarrow \sum_l \left.-(l+1)\frac{B_l}{r^{l+2}}\right|_{r=R} P_l - \sum_l l A_l \left. r^{l-1}\right|_{r=R} P_l = -\frac{\sigma_0}{\varepsilon_0}$$

If know σ_0, then 2nd eq'n tells you about

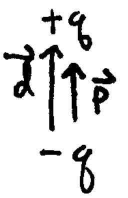
$$-\frac{(l+1)B_l}{R^{l+2}} - l A_l R^{l-1} = \left(\text{whatever the } l^{th} \text{ coeff of } -\frac{\sigma_0}{\varepsilon_0} \text{ is}\right)$$

2 eq'ns in 2 unknowns A_l, B_l!

("Done")

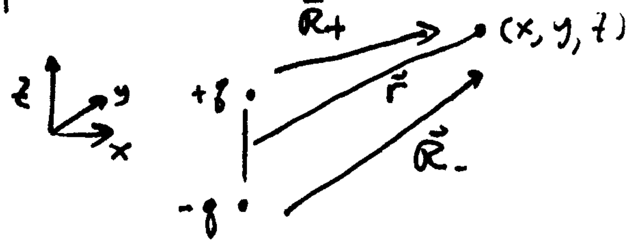
Phys 3310 - 3-33

Dipoles & Multipoles



This is an electric dipole: 0 net charge.

we define $\vec{p} = q\vec{d}$ = "the dipole moment". (We'll see why right away)



• Consider $V(\vec{r}) = \dfrac{q}{4\pi\varepsilon_0}\left[\dfrac{1}{R_+} - \dfrac{1}{R_-}\right]$

with $R_\pm = \sqrt{x^2+y^2+(z\mp\frac{d}{2})^2} = \sqrt{r^2 \mp dz + d^2/4}$ ← Approximation!

If $d \ll r$, $R \simeq r\sqrt{1 \mp \frac{dz}{r^2}} + \ldots \simeq r\left(1 \mp \frac{1}{2}\, dz/r^2\right)$

so $V(\vec{r}) \simeq \dfrac{q}{4\pi\varepsilon_0}\left[\dfrac{1}{r}\left(1+\dfrac{1}{2}\dfrac{dz}{r^2}\right) - \dfrac{1}{r}\left(1-\dfrac{1}{2}\,dz/r^2\right)\right] \simeq \dfrac{q\,dz}{4\pi\varepsilon_0 r^3}$

But note that $\vec{p}\cdot\vec{r} = p_z \cdot r_z = (qd)(z)$ so.

$$V(\vec{r}) \simeq \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$$

Not coulomb, it drops like $\frac{1}{r^2}$, not $\frac{1}{r}$.

→ Note that this is "coordinate free", we can now choose any axes we like!

• Consider $\vec{E}(\vec{r}) = -\vec{\nabla} V$

Moving to spherical coords $V(\vec{r}) = \dfrac{p\cos\theta}{4\pi\varepsilon_0 r^2}$

$$E_r = -\frac{\partial V}{\partial r} = +\frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$$

$$E_\theta = \frac{-1}{r}\frac{\partial V}{\partial\theta} = +\frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

so $$\vec{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

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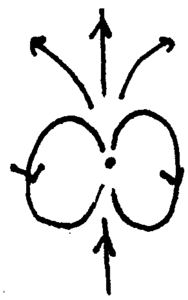
$\vec{E}$ sketch

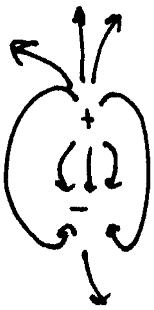


Note: $\theta = \pi/2 \Rightarrow \vec{E} = \frac{p}{4\pi\epsilon_0 r^3}(\hat{\theta})$ points "down" dies fast.

$\theta = 0 \Rightarrow \vec{E} = \frac{p}{4\pi\epsilon_0 r^3} \cdot 2\hat{r}$ " up.

It's an <u>ideal</u> or <u>pointlike</u> dipole. ($d \to 0$, but $p = qd$ is finite)

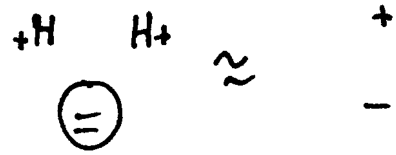
<u>Real</u> Dipole



Looks like above, (+ looks a <u>lot</u> like it far away!)
(But <u>not</u> right up close)
(Need more terms in that expansion!)

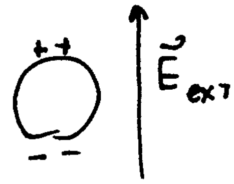
<u>Where do Dipoles come from?</u>

1) <u>Natural</u> (polar molecules) e.g. H_2O :



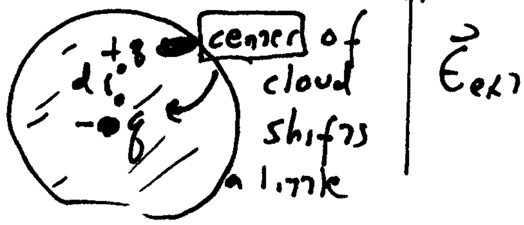
(Typical "d" = ~Angstroms ~10^{-10} m
"q" = e, so $p \simeq 10^{-26}$ C m)

2) <u>INDUCED</u> : Put neutral object in an $\vec{E}$ field,



think of <u>atom</u>

$\Rightarrow$



~~How much shift? Δ.11~~ $\cancel{\frac{q^2}{4\pi\epsilon_0 d^2}}$ ~~attraction~~ $\cancel{= q\vec{E}_{ext}}$ ~~"repulsion"~~

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so, at equilibrium, pull on $+q$ to center = push from $\vec{E}_{ext}$

NOT $\dfrac{q^2}{4\pi\epsilon_0 d^2}$! Need field inside "sphere of charge", which by Gauss' law is $\underset{\text{uniform sphere}}{E} = \dfrac{q\,r}{4\pi\epsilon_0 R^3}$ ← remember this? If not, work it out!

so $\cancel{q}\dfrac{q\,d}{4\pi\epsilon_0 R^3} = E_{ext}\,\cancel{q}$

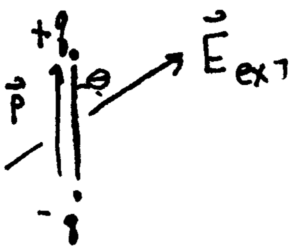
which means $p = q\,d = 4\pi\epsilon_0 R^3\, E_{ext}$.

- Polarization is linear in E_{ext}!
- Estimate "polarizability" $\equiv P/E \equiv \alpha \simeq 4\pi\epsilon_0 R^3$ ("size of atom", about 10^{-10} m.)

works pretty well, see Griffiths p. 161!

Once you have a dipole: It creates $\vec{E}$ field
" has nonzero $V(r)$ ⇒ potential energy associated w/ it
It can create torque on other dipoles!

$\boxed{\text{Even though it's neutral, and "tiny".}}$ It feels a torque if put it in an E-field, + Force too!



$$\vec{\tau} = \sum \vec{r}\times\vec{F} = 2\, q E_{ext}\frac{d}{2}\sin\theta$$
about origin

$$= q\,d\,E_{ext}\sin\theta \quad \text{into the page}$$

$$= \vec{p}\times\vec{E}$$

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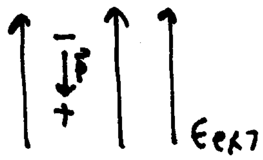
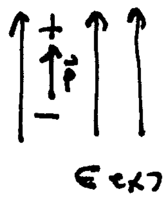
It has **energy** associate with orientation ~~θ~~ in external field



High energy



Lower energy

$$U = \underbrace{-\vec{p}\cdot\vec{E}}$$

lowest when $\vec{p}$ points in $\vec{E}$ direction

It feels **force** if $\vec{E}$ bigger on one side than other,

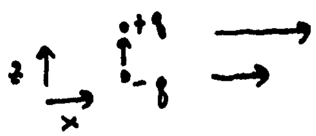
$$\vec{F}_{\text{net on dipole}} = (\vec{p}\cdot\vec{\nabla})\vec{E}$$

For a Dipole $\uparrow$ in z direction,

$$\vec{F}_{\text{net on dipole}} = p\frac{\partial}{\partial z}\vec{E} = \left(p\frac{\partial E_x}{\partial z},\ p\frac{\partial E_y}{\partial z},\ p\frac{\partial E_z}{\partial z}\right)$$

(yuck!)

Ex:



Here $\frac{\partial E_x}{\partial z} > 0$, so feels $\vec{F}_{net}$ in $\underline{\underline{x}}$ direction!

(**Also** feels $\vec{\tau} = \vec{p}\times\vec{E}$)

Let's go back to dipole as **source**:

We found $V_{dipole}(\vec{r}) = \dfrac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}$ ← dies like $\frac{1}{r^2}$. [Simple "pattern", $\vec{p}\cdot\hat{r} = p\cos\theta$ for $\vec{p}\uparrow = p\hat{z}$, e.g.]

Recall $V_q(\vec{r}) = \dfrac{q}{4\pi\epsilon_0 r}$ ← dies like $\frac{1}{r}$. [**No** "pattern", same at all θ]

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(+q) = "Monopole". $V \sim 1/r$ → (only exactly true if q is pointlike + spherically symm)

(+q) (−q) = "Dipole" $V \sim 1/r^2$ (simple angular dependence)

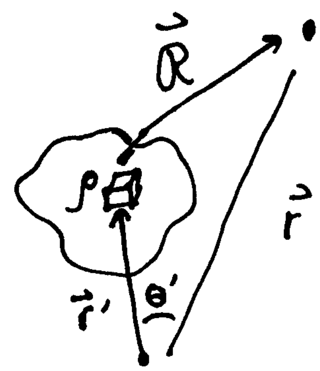
only exactly true if +q −q are very close

Leads us to a method to estimate/approximate $V(r)$ when get far from a bunch of charges.

[If $q_{net} \neq 0$, $V \sim 1/r$ "dominates".]

Multipole Expansion.

Go back to $V(r) = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(r')\,d\tau'}{\mathscr{R}}$



If $\vec{r} \gg$ all $\mathscr{R}$'s, "far away", this complex $\rho(r')$ looks simple.

(like a point charge! or, if $q_{net} = 0$, like a dipole.

or, if e.g. $\pm$ $\mp$ like this, two canceling dipoles, it's a "quadrupole", and $V \sim \frac{1}{r^3}$)

we can be more explicit:

$$\mathscr{R}^2 = r^2 + r'^2 - 2rr'\cos\theta' = r^2\left(1 + \frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta'\right)$$

pull this out, it's big, (or will be ...)

~~$\mathscr{R} = r\sqrt{(1+\varepsilon)}$ with $\varepsilon = \frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta$ (will be small)~~

Let's call $\varepsilon = \frac{r'}{r}$, which will be small when we're <u>far</u> from the charge distribution

$$\mathcal{R} = r\sqrt{1 - 2\varepsilon\cos\theta' + \varepsilon^2} \qquad \left(\text{and we'll need } \frac{1}{\mathcal{R}}\right)$$

Recall $$(1+\eta)^{-1/2} = 1 + \frac{(-1/2)}{1!}\eta + (-\tfrac{1}{2})(-\tfrac{1}{2}-1)\frac{\eta^2}{2!} + (-\tfrac{1}{2})(-\tfrac{1}{2}-1)(-\tfrac{1}{2}-2)\frac{\eta^3}{3!} + \dots$$

so $$\frac{1}{\mathcal{R}} \equiv \frac{1}{r}(1 - 2\varepsilon\cos\theta' + \varepsilon^2)^{-1/2}$$

$$\simeq \frac{1}{r}\left(1 - \frac{1}{2}\left[-2\varepsilon\cos\theta' + \varepsilon^2\right] + \left(\frac{3}{8}\right)\left[-2\varepsilon\cos\theta' + \varepsilon^2\right]^2 + \dots\right.$$

collect all terms with ε, ε^2, etc

$$\approx \frac{1}{r}\left(1 + \varepsilon\cos\theta' - \frac{1}{2}\varepsilon^2 + \frac{3}{8}\cdot 4\cos^2\theta'\,\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right)$$

$$\simeq \frac{1}{r}\left(1 + \varepsilon\cos\theta' + \left(-\frac{1}{2} + \frac{3}{2}\cos^2\theta'\right)\varepsilon^2 + \dots\right)$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(r')\,d\tau'}{r}\left(1 + \varepsilon\cos\theta' + \left(-\frac{1}{2} + \frac{3}{2}\cos^2\theta'\right)\varepsilon^2 + \dots\right)$$

Look at the terms one at a time — leading — smaller — smaller still

Leading: $$V(r) \approx \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{r}\underbrace{\iiint \rho(r')\,d\tau'}_{\text{This is just } Q_{TOT}.}$$

came out of integral!

so $V(r) \simeq \frac{Q}{4\pi\varepsilon_0 r}$, the "monopole term", just potential of <u>point</u> charge.

ε

<u>Next</u> term: $\frac{1}{4\pi\varepsilon_0}\iiint \rho \frac{(r')d\tau'}{r}\left(\frac{r'}{r}\cos\theta'\right)$

$$= \frac{1}{4\pi\varepsilon_0 r^2}\underbrace{\iiint \rho(r')r'\,d\tau'\cos\theta'}$$

drops off <u>faster</u>. ←

Note: This integral is a <u>number</u>, independent of $\vec{r}$. It's a <u>property</u> of the charge distribution! It's called "the dipole moment".

Note: $\cos\theta' = P_1(\cos\theta')$ is the first Legendre polynomial.

<u>Next</u> term: $\frac{1}{4\pi\varepsilon_0}\iiint \rho \frac{(r')d\tau'}{r}\cdot\frac{r'^2}{r^2}\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\underbrace{\iiint \rho(r')\,d\tau'\; r'^2\cdot P_2(\cos\theta')}_{\text{the "quadrupole moment"}}$$

And so it goes...

$$V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_{tot}}{r} + \frac{\text{Dipole moment}}{r^2} + \frac{\text{Quad moment}}{r^3} + \dots\right)$$

- when r is big, <u>leading</u> term dominates.

  so e.g. If $Q_{tot} = 0$, Dipole term ".

- Note that $\cos\theta'$ appears, which <u>does</u> involve relative direction of $\vec{r} + \vec{r}'$, so these <u>moments</u> <u>do</u> <u>depend</u> on location/orientation of ρ and $\vec{r}$ (but not on $|r|$)

<u>Nice</u>: we can approximate $V(r)$ far away without fussing about the details, <u>one</u> <u>number</u> will likely tell us how $V(r)$ behave far away.

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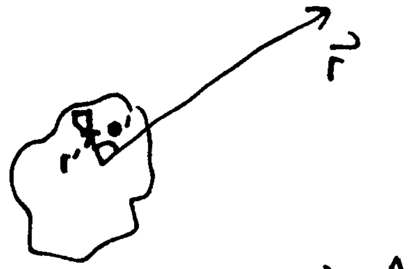
Let's look at the Dipole term. (e.g. imagine $\underline{\underline{Q_{tot} = 0}}$)



$$V_{Dip}(r) \approx \frac{1}{4\pi\varepsilon_0 r^2} \iiint \rho(r')\, r' \cos\theta'\, d\tau'$$

$$\approx \frac{1}{4\pi\varepsilon_0 r^2} \iiint \rho(r')\, \vec{r}\,' \cdot \hat{r}\, d\tau'$$

Note: $r'\cos\theta' = \vec{r}\,' \cdot \hat{r}$

$$= \frac{1}{4\pi\varepsilon_0 r^2} \left(\iiint \rho(r')\, \vec{r}\,' d\tau' \right) \cdot \hat{r}$$

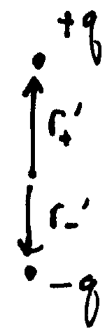
$\underline{\underline{\text{recall}}}$ $\quad V_{\underline{\text{pure}}\,\text{dip}} = \dfrac{1}{4\pi\varepsilon_0 r^2}\, \vec{p} \cdot \hat{r}$

p. 6

$$\boxed{\text{In } \underline{\text{general}},\quad \vec{p} = \text{"dipole moment"} \equiv \iiint \rho(\vec{r}\,')\, \vec{r}\,' d\tau'}$$

For  Point charges: $\displaystyle \iiint q\, \delta^{(3)}(r_+')\, \vec{r}\,' d\tau' + \iiint -q\, \delta^{(3)}(\vec{r}_-')\, d\tau'$

$$= q\, \vec{r}_+\,' + (-q)\, \vec{r}_-\,'$$

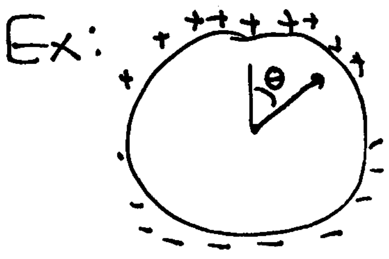


$$\vec{p} = \sum_i q_i \vec{r}_i\,'$$

In our case, $\underline{\text{pure dipole}}$, $\quad q\,\frac{d}{2}\hat{z} + -q\left(-\frac{d}{2}\hat{z}\right) = q d\, \hat{z} = \vec{p}$ ✓

Bottom line: the dipole term, for a $\underline{\text{dipole}}$, is $\underline{\text{everything}}$! :)

The dipole moment of $\underline{\underline{\text{any}}}$ distribution can be computed.

Ex: 

$\underline{\text{shell}}$, $\rho = \sigma_0\, \delta(r-R) \cdot \cos\theta$ $\quad$ has a dipole moment:

$$p_z = \iiint \rho(\vec{r}\,')\, z'\, d\tau' = \iiint \sigma_0\, \delta(r'-R) \cos\theta' \cdot r'\cos\theta' \cdot r'^2 \sin\theta'\, d\theta'\, d\varphi'$$

$$= \sigma_0 \cdot R^3 \, \frac{-\cos^3}{3}\Big|_0^{\pi} = \frac{2}{3}\sigma_0 R^3$$

(Convince yourself, ϕ integral kills off p_x and p_y.)

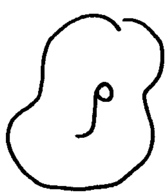
Summary: Finite charge distributions can be characterized by numbers, "moments".
(which depend on direction you're "looking from")

Then $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\Big[\underbrace{\frac{Q}{r}}_{\text{Monopole term}} + \underbrace{\frac{\vec{p}\cdot\hat{r}}{r^2}}_{\text{Quadrupole term}} + \frac{\text{Quadrupole term}}{r^3} + \dots\Big]$$

these moments are easy to compute, given ρ.
The dipole moment $\vec{p}$ is just what it should be, $\sum_i q_i \vec{r}_i$

we talked about "dipole moments" of neutral molecules earlier,
so the $V(\vec{r})$ will depend on $\underline{\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}}$ + higher order terms that drop off faster

so $\vec{p}$, a "property of the dipole" (like the charge, but a vector) is the key information we need to know $V(r)$ (+ thus $\vec{E}$, and forces, etc)

In general:  has a dipole moment
$$\vec{p} = \iiint \rho(\vec{r}\,')\,\vec{r}\,'\,d\tau'$$

(It depends on your choice of origin ... unless $q_{tot} = 0$, then it turns out, (like $\vec{p} = q\vec{d}$) to be "universal".)