Big Picture: we'd like to know $\vec{E}$, given charges!
[why? $\vec{F} = q\vec{E}$, so we can predict/describe/control motion!]

Method 1: $\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\, d\tau}{R^2}\hat{R}$ ← Straightforward but often hard in practice

Especially problematic if ρ "adjusts", like charges on a conductor, you might not know ρ everywhere...

Method 2: Given V, then $\vec{E} = -\vec{\nabla} V$

OK, where's V come from? $V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\, d\tau}{R}$ ← Easier, but still often hard

(And the story when ρ adjusts is still an issue)

Method 3: Get V from Diff Eq: $\boxed{\nabla^2 V = -\rho/\varepsilon_0}$ ← "Poisson"

and look for V in regions where no charge (!)

So, we will need "boundary conditions"

⇒ you tell me about V at the boundary, and then inside $\nabla^2 V = 0$ ← Laplace's eq'n

will tell me V (and thus $\vec{E} = -\vec{\nabla} V$)

3310 3-2

LAPLACE: $\nabla^2 V = 0$, or $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

MANY places this shows up in physics!

- Heat flow $\rightarrow$ $\nabla^2 T = 0$ (for time independ. situations)
- Hydrodynamics
- Diffusion.
- ~~Travelling waves~~

• Methods to solve it are general (+ can then, often, be applied to e.g. Poisson's eq'n if you need that)

• Might think $\rho = 0$ is limited, but very often you do want V (+ $\vec{E}$) in space, given charges or conductors at "walls".

• Methods will also be used in Quantum mechanics, Plasma Physics, Travelling waves

[e.g. $-\frac{\hbar^2}{2m}\nabla^2\psi = (E-V)\psi$ different, but related.]

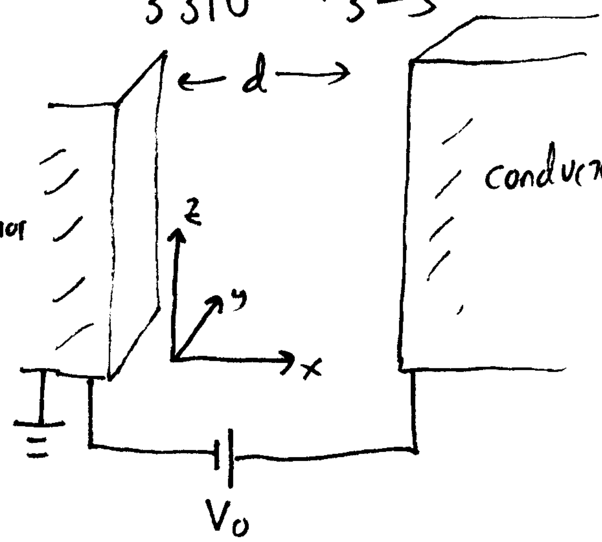
(Spherical:) $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$

• Remember: $V(\vec{r})$ is a scalar.

V = constant in and on conductors.

There are some cool math theorems that will help us solve + use $\nabla^2 V = 0$ ...

Ex:



This is a 1-D problem, because V clearly cannot depend on y or z

$V(x,y,z) = V(x)$

In the gap, $\nabla^2 V = 0$, $\Rightarrow \dfrac{d^2 V(x)}{dx^2} = 0$

At boundary, we know $V(0) = 0$, $V(d) = V_0$.

[There's our boundary condition. I don't know ρ there, but don't care either!]

Soln: $\dfrac{d}{dx}\left(\dfrac{dV}{dx}\right) = 0 \Rightarrow \dfrac{dV}{dx} = C$ ← constant

$\Rightarrow V = Cx + D$ // General soln

But $V(0) = 0 \Rightarrow D = 0$

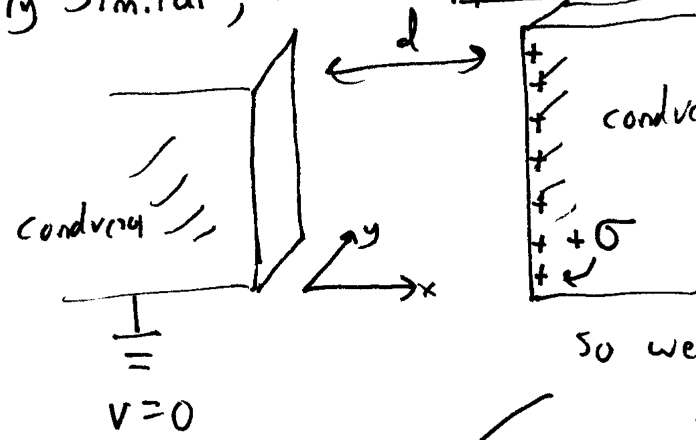
$V(d) = V_0 \Rightarrow C = V_0/d$, so $V(x,y,z) = \dfrac{V_0 x}{d}$.

(This is a capacitor.)

Note: V is smooth, simple, boring. It has MAXIMA only at edges

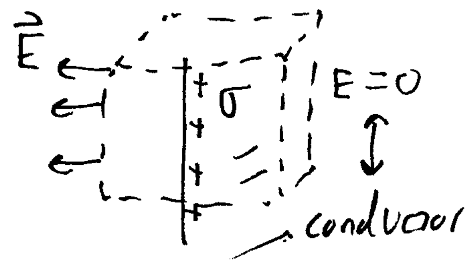
It has its average value right in middle.

Try Similar, but different B.C.: Give you σ, rather than V. at right edge.



(we have dumped charge, and **measure** $+\sigma$ on ~~[illegible]~~ wall)

so we know $\vec{E}$ just outside conductor (i.e. $\vec{E}(x=d)$)



(No flux on 5 faces, only flux@ left face!)

Gauss' law says

$$0+0+0+0+0+E \cdot \text{Area} = \frac{\sigma \cdot \text{Area}}{\varepsilon_0}$$

$\Rightarrow \vec{E} = \sigma/\varepsilon_0 \, (-\hat{x})$ (just at ~~[illegible]~~ boundary, $x=d$)

so $\vec{\nabla} V = -\vec{E}$ tells us $\left.\frac{dV(x)}{dx}\right|_{x=d} = +\sigma/\varepsilon_0$

so $V = Cx + D$, $V(0) = 0$

so $V = Cx$, and $\frac{dV}{dx} = C$ tells us

$$V = \frac{\sigma}{\varepsilon_0} x$$

- If left wall (conductor) is at ~~[illegible]~~ $V(0) = V_0$

then $V = \frac{\sigma}{\varepsilon_0} x + V_0$.

3310 3-5

- General properties of sol'ns of $\nabla^2 V = 0$:

(Provable, but let's just use them!) (True in 3-D !)

① V has no "local maxima or minima" anywhere but boundaries. (Proved in Griff, p.114)

② V is smooth (Griffiths calls it "boring") + continuous everywhere

③ V(center) = average of V over any surrounding sphere

$$V(\vec{r}) = \frac{1}{4\pi R^2} \underbrace{\oint\!\!\oint V \, dA}_{\text{sphere centered on } \vec{r}}$$ (Proved in Griff, p.114)

④ V(r) is unique: If $\nabla^2 V = 0$, and you have boundary conditions – Either value of V at b'dry or $\frac{\partial V}{\partial n}$ at b'dry ← "normal" to b'dry

then sol'n is unique.

Consequence of ④ If you can guess a sol'n which satisfies $\nabla^2 V = 0$ and V(boundary conditions), you're done!!

We'll use this a lot! [Note: It's true for Poisson's eq'n too!]

Consequence of (3) Method of relaxation.

If you know $V(r)$ on boundary then

- **Guess** $V(r)$ on a "grid" of points inside
- Step through all interior points, resetting $V(r) =$ average of surrounding pts
- Repeat ....

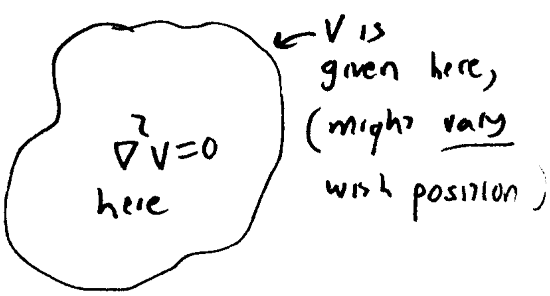
you will settle on a numerical approximation

very useful in practice, (but we won't do this in this class)

But nice to know you can always solve $V(\vec{r})$ numerically

---

Proof of uniquess if V is specified on b'dry



Suppose there are two different V's, $\left[\begin{array}{l}\nabla^2 V_1 = 0 \\ \nabla^2 V_2 = 0.\end{array}\right.$, $\left[\begin{array}{l}V_1(\text{b'dry}) = V_2(\text{b'dry}) \\ \quad = V_{given}(\text{b'dry})\end{array}\right]$

Trick: $\nabla^2 (V_1 - V_2) = \nabla^2 V_1 - \nabla^2 V_2 = 0.$

But $V_1 - V_2 = 0$ everywhere on boundary. So

fact (1) (V has maximum on b'dry) says $V_1 - V_2 = 0$ everywhere
has minimum

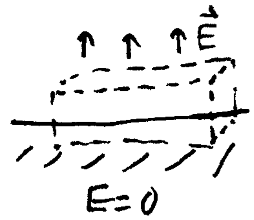
Q.E.D.

- $\underline{\underline{\text{Boundary conditions}}}$: Either V $\underline{\underline{\text{or}}}$ $\frac{\partial V}{\partial n}$ is sufficient, (to find V inside region)

(or a mix!) !

See notes, Ch. 2 p. 35 (Griffiths pp. 88–89)

① Near any conductor

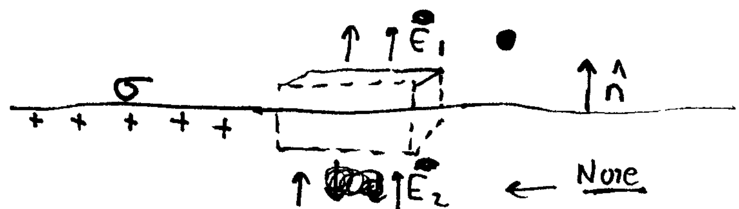


$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$\hat{n}$: normal.

which means $\frac{\partial V}{\partial n} = -\frac{\sigma}{\varepsilon_0}$ (or $\vec{\nabla} V \cdot \hat{n} = -\sigma/\varepsilon_0$)

② Near any sheet of charges



← Note E_2 might be +/−

$$E_1 \cdot A - E_2 \cdot A = \frac{\sigma A}{\varepsilon_0}$$

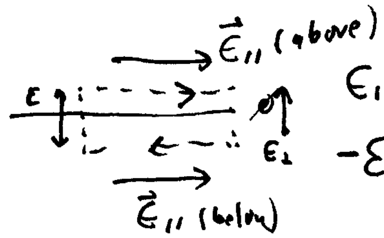
↑ Given my arrow choice in fig, i.e. $\vec{E}_{above} = E_1 \hat{n}$

$\vec{E}_{below} = E_2 \hat{n}$

so $E_{\substack{normal \\ above}} - E_{\substack{normal \\ below}} = \sigma/\varepsilon_0$

thus $\left.\frac{\partial V}{\partial n}\right|_{above} - \left.\frac{\partial V}{\partial n}\right|_{below} = -\sigma/\varepsilon_0$

③ $\vec{\nabla} \times \vec{E} = 0$ says



$E_{\parallel}$ above $\cdot L - E_\perp^{(right)} \cdot \varepsilon$

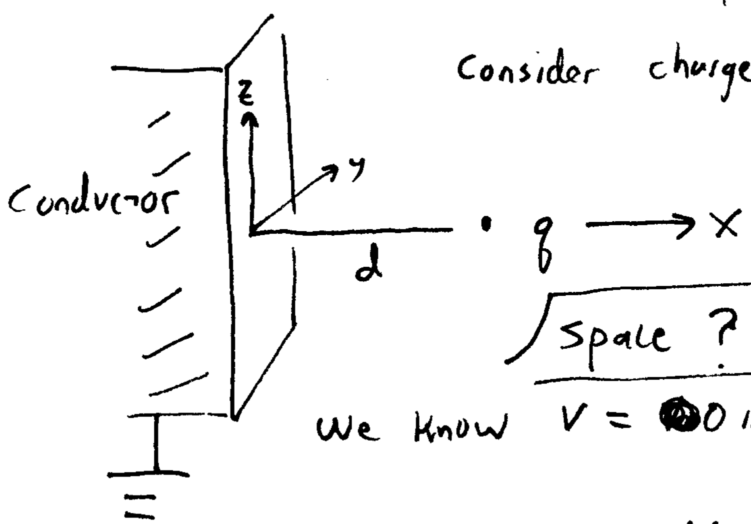
$- E_{\parallel}$ below $\cdot L + \underbrace{E_\perp (\text{left}) \cdot \varepsilon}_{\text{small!}}$ $= 0$

so $E_\parallel$ above $= E_\parallel$ below ← $E_\parallel$ is $\underline{\underline{\text{continuous}}}$

④ $V_{above} = V_{below}$ (always continuous!)

Method of IMAGES:

- Remember, if ~~[illegible]~~, $\nabla^2 V = \rho/\varepsilon_0$ in a region, and if we can guess a V which obeys ↑, and which is correct at boundary of our region, we're done, we have $V(\vec{r})$!



Consider charge q, distance d from ∞ sheet of conductor

What is $V(\vec{r})$ throughout all space? (Not simple, 'cause you induce a complicated σ on sheet!)

We know $V = 0$ in conductor. (Grounded, equipotential!)

We know $V = 0$ off at ∞. or off at $r = \infty$

so we know $\underset{=0}{V}$ on giant boundary (wall on left, everywhere else)

and we want V in rest of space. (the "right half" of universe)

We will use a trick. I can create an imaginary scenario

where $V = 0$ everywhere on $x = 0$ plane
$V = 0$ off at ∞

and $\nabla^2 V = \rho/\varepsilon_0$ in "right half", and $V(r)$ is easy to compute.

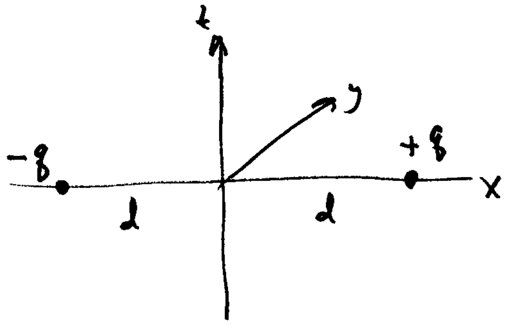
If I do this, we're done. By uniqueness, that is the unique sol'n

It satisfies all our boundary conditions and Poisson's eq'n.

(And, we won't need to know the complicated σ on the sheet which nature generates to cancel $\vec{E}$ out to 0 inside conductor)

3310 3-9

The trick: Consider <u>this</u> problem:



<u>No</u> conductor.

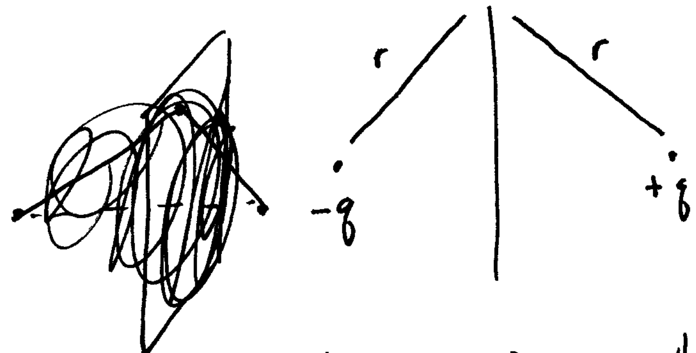
<u>Nothing</u> but a new (made up) "image charge" at $x = -d$.

1) $V(r)$ here is just $\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\vec{r} - d\hat{x}|} + \frac{-q}{|\vec{r} + d\hat{x}|}\right)$ by inspection

2) $V(\infty) = 0$. Good!

3) V (on yz plane) $= 0$.

Distance is <u>same</u> to $\pm q$!



$\nabla^2 V(r) = \rho/\varepsilon_0$, for <u>right</u> half of universe, (this is 0 everywhere except $q\,\delta^{(3)}(\vec{r} - d\hat{x})$, it's <u>perfectly</u> fine)

So for <u>right</u> <u>half</u> of universe our V has correct B.C.

+ solves Poisson. So we're done.

That's it!

$$V(x,y,z)\Big|_{x>0} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+d)^2+y^2+z^2}}\right)$$

Important comments.

① This V is <u>dead</u> <u>wrong</u> for $x<0$ (where $V=0$, right?!)

② There are only a limited # of situations where this method works (e.g. conducting sphere + conducting sheet + cond. cylinder)

<u>But</u> important for people who think about antennas!

③ The <u>method</u> is: make up a <u>new</u> situation, where you

Add image charge(s) in special spots (with special q's)

such that
- Images are not located in the region of space you want to know $V(r)$ for (!)
- V at <u>boundaries</u> of region you're interested in is same as your (real) situation

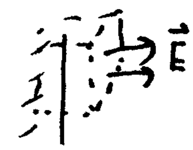
And, done! $V = \sum_i \frac{1}{4\pi\varepsilon_0} \frac{q_i}{|\vec{r} - \vec{r}_i|}$ easy to write down!

<u>only</u> valid in the region you were considering, <u>not</u> all space.

Some q_i's are <u>real</u>, some are fictitious.

- So, this method works if have <u>bunch</u> of q's outside sheet (just add <u>bunch</u> of matching image charges!)

By the way, since we know $V(x>0, y, z)$, we can easily figure out σ on the sheet!



Remember, $\vec{E} = \frac{\sigma(y,z)}{\varepsilon_0}\hat{x}$ just outside the conductor

But $\vec{E} = -\vec{\nabla} V$, so $\frac{\sigma}{\varepsilon_0} = -\frac{\partial V}{\partial x}\Big|_{x=0}$

$$\boxed{\text{Griffiths: } \frac{\sigma}{\varepsilon_0} = -\frac{\partial V}{\partial n}\Big|}$$

This derivative is not so hard, $\left[\frac{\partial}{\partial x}(\ \ )^{-1/2} = -\frac{1}{2}\cdot(\ \ )^{-3/2}\cdot\frac{\partial}{\partial x}(\text{inside})\right]$

$$\frac{\sigma(y,z)}{\varepsilon_0} = -\frac{q}{4\pi\varepsilon_0}\left(\frac{-(x-d)}{((x-d)^2+y^2+z^2)^{3/2}} + \frac{(x+d)}{((x-d)^2+y^2+z^2)^{3/2}}\right)\Bigg|_{x=0}$$

$$= \frac{-qd}{4\pi\varepsilon_0}\left(\frac{2}{(d^2+y^2+z^2)^{3/2}}\right)$$

- It peaks at $y=z=0$ + then "fades out" ✓
- It's negative everywhere ✓ (It would've been hard to guess!)
- Griffiths shows $\iint \sigma(y,z)\,dy\,dz = -q$. ✓

So our fictitious image charge, $-q$, is physically manifested by $-q$ "smeared" appropriately on surface of conductor.

(There really is not any "$-q$" at $x=-d$!!)

(• oh, one more thing – If you try e.g. $E_y = -\frac{\partial V}{\partial y}$, you'll get 0 at $x=0$, check for yourself!)

3310 3-12 Notes (opt)

Work + Energy:

$$W_{\substack{\text{2 charges in} \\ \text{"fictitious" situation}}} = \frac{1}{4\pi\varepsilon_0}\frac{(+q)(-q)}{(2d)}$$

$$W_{\substack{\text{real situation, +q} \\ \text{outside grounded plane}}} = \frac{1}{2}\varepsilon_0 \underbrace{\iiint}_{\text{only right half of universe contributes!}} E^2\, d\tau = \frac{1}{2} W_{\substack{\text{fictitious} \\ \text{world}}}$$

Why not same? See Griffiths... but here's my explanation

(i) In "fictitious world", you bring $-q$ to $x=-D$ (no work!) + then bring $+q$ to $+D$, doing (−) work all the way.

(ii) In "real world", you bring $+q$ to $+D$, and it "sees" an image charge which is always farther away than what it would be in scenario i, with image fixed at $-d$. (Because image is always at "$-x$")

- Note that, when q is at $x=+d$, the force on it (at that spot) is just $\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}(-\hat{x})$. Because $\vec{E}(x)$ is same whether in real world or "image charge" world, since $\vec{E} = -\vec{\nabla} V$.

3310 Notes 3-13

Separation of Variables.

We just solved Poisson's eq'n for some very special cases (∞ grounded conducting sheet, e.g). But it's not general enough, we want to tackle $\nabla^2 V = 0$, given boundary conditions, for more circumstances, i.e. more generally....

The idea is this: $V(x,y,z)$ could be complicated! It might e.g. be $\sim \frac{1}{\sqrt{x^2+y^2+z^2}}$. But in some (many!) circumstances, we will find that $V(x,y,z) =$ (fn of x) * (another fn of y) * (another fn of z)

Like, say $e^x \cdot \cos(y) \cdot \sin(z)$ or something.

And if it isn't, it might still be some combination of such fns,

Like say $e^x \cos y \sin z + e^{2x} \cos y \sin z + e^x \cos 2y \sin 3z + \ldots$

↳ In fact, we can pretty much build $\boxed{\text{any}}$ function V up like this!)

So, we'll see that if $V(x,y,z) = X(x)\, Y(y)\, Z(z)$,

(or any sum of such fns) then we can solve $\nabla^2 V = 0$

in many many situations!

- This method will only help us if we can then "match our boundary conditions", but is quite general + powerful.

3310 Notes 3-14

Bottom line for "sep. of variables"

we have $\nabla^2 V = 0$ and given B.C.'s

we <u>try</u> $V(x,y,z) \stackrel{?}{=} \mathcal{X}(x)\,\mathcal{Y}(y)\,\mathcal{Z}(z)$. Just try it!

<u>If</u> it works, and <u>if</u> it satisfies our BC's, uniqueness $\Rightarrow$ <u>done</u>.

(+ If it fails, but a <u>sum</u> of such fns " ", " $\Rightarrow$ <u>done</u>)

↳ use superposition.

By the way, it might work better to try, e.g.

$$V = R(r)\,\Theta(\theta)\,\Phi(\phi) \quad \text{(in spherical coords!)}$$

we'll get there soon :)

OK, so $\nabla^2 V = \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$

<u>Try</u> $V = \mathcal{X}(x)\mathcal{Y}(y)\mathcal{Z}(z)$, and <u>note</u> e.g. $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{d^2 \mathcal{X}(x)}{dx^2}\mathcal{Y}(y)\,\mathcal{Z}(z)$

Partial deriv $\Rightarrow$ <u>Total</u> deriv (do you see why?) * No x-dep, they're constants

so we have $\mathcal{X}''(x)\,\mathcal{Y}(y)\,\mathcal{Z}(z) + \mathcal{X}(x)\,\mathcal{Y}''(y)\,\mathcal{Z}(z) + \mathcal{X}\,\mathcal{Y}\,\mathcal{Z}''(z) = 0$

where $\mathcal{X}''$ means $\dfrac{d^2}{dx^2}$. Next, <u>divide</u> <u>both</u> <u>sides</u> by $\mathcal{X}(x)\mathcal{Y}(y)\mathcal{Z}(z)$

3310 Notes 3-15

So we have a lot of cancellation (do it, check!)

$$\frac{\mathcal{X}''(x)}{\mathcal{X}(x)} + \frac{\mathcal{Y}''(y)}{\mathcal{Y}(y)} + \frac{\mathcal{Z}''(z)}{\mathcal{Z}(z)} = 0.$$

pure function of x + pure fn of y + pure fn of z = 0.

Now you must convince yourself that this is nuts!

How could $f(x) + g(y) = 0$ for all x and y?!

No way!! Vary x, with y fixed, + you'll change the sum...

unless $f(x)$ is just a constant, i.e. it really doesn't depend on x at all!

Conclusion: $\nabla^2 V = 0$ and $V = \mathcal{X}(x)\mathcal{Y}(y)\mathcal{Z}(z)$ requires

$$\frac{\mathcal{X}''(x)}{\mathcal{X}(x)} = C_1 \qquad \frac{\mathcal{Y}''(y)}{\mathcal{Y}(y)} = C_2 \qquad \frac{\mathcal{Z}''(z)}{\mathcal{Z}(z)} = C_3$$

with $C_1 + C_2 + C_3 = 0$.

We have three simple 2nd order ordinary diff eq's to solve.

Consider $\mathcal{X}''(x) = C_1 \mathcal{X}(x)$. Look familiar! It has quite simple sol'ns, (depending on the sign of C_1, only!)

$\cancel{\mathcal{X}(x) = A\cos(x) + B\sin x}$

3310 Notes 3-16

2 undetermined constants

$X''(x) = C_1 X(x)$. General sol'n is $X(x) = Ae^{\sqrt{C_1}x} + Be^{-\sqrt{C_1}x}$

check for yourself!

If $C_1 > 0$, $X(x) = Ae^{\sqrt{C_1}x} + Be^{-\sqrt{C_1}x}$

If $C_1 < 0$, get complex EXPONENTIALS, which you ~~may~~ can think of as combo's of sin + cos by Euler's theorem

$\hookrightarrow X(x) = A'\sin(\sqrt{-C_1}\,x) + B'\cos(\sqrt{-C_1}\,x))$

If $C_1 = 0$, $X(x) = A'' + B''x$.

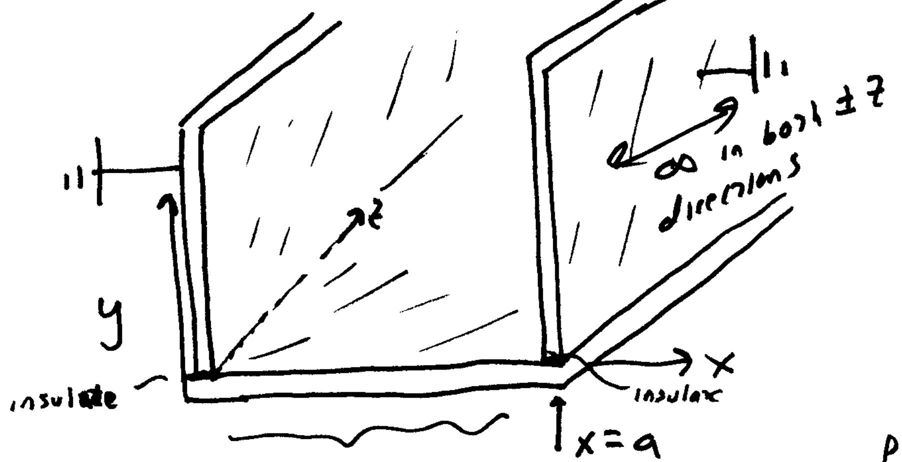
So we have traded our Partial Diff Eq for 3 (easy) ODE's, at a cost of lots of undetermined constants popping up.

Our boundary conditions will have to determine all those!

At this point we just need to get concrete! Let's start with a 2-D problem to warm up. This just means $V = V(x,y)$, it has no z-dependence. Physically, we just need a setup which is uniform in z, so nothing will vary in that direction.

3310 Notes 3-17

Ex: Consider a metallic (square) ~~pipe~~ gutter which extends in $\pm z$ directions, forever.



"hot" part: Voltage here = $V_0(x)$

Lets ground the 2 side walls $(V=0)$ but insulate the base, and allow it to have some other potential, maybe even $V_0(x)$

of course, if this base is a conductor, $V_0(x) = V_0 =$ constant

(But maybe the base is not a conductor, or e.g. long strips of conductor separated by thin insulators?)

Inside the pipe (i.e $0 \le x \le a$, $y > 0$, any z) is empty so $\nabla^2 V = 0$ there.

Boundary conditions

$V(x=0, y>0) = 0$ grounded!

$V(x=a, y>0) = 0$ grounded

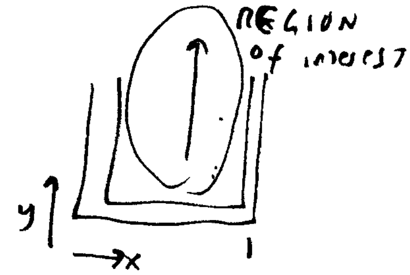
$V(\text{any } x \text{ from } 0 \to a, y=0) = V_0(x)$ fixed voltage, somehow.

$V(\text{any } x, y=\infty) = 0$ ← physically reasonable,

at $y=\infty$ we're far from the "hot" sheet, $V \to 0$

Note: No B.C. on z, but V can't depend on z by symmetry!
So it's really a 2-D problem!

3310 Notes 3-18



- Set to go: we have region with $\nabla^2 V = 0$
  and we know V at all 4 boundaries!
  (zero on left, right, and top (at infinity)), finite but given at bottom.

Try $V(x,y) = \mathcal{X}(x)\,\mathcal{Y}(y)$. Separate variables

so $\mathcal{X}''(x) = C_1 \cdot \mathcal{X}(x)$

$\mathcal{Y}''(y) = C_2\, \mathcal{Y}(y)$ but $C_1 + C_2 = 0$, so $C_2 = -C_1$

which is positive, C_1 or C_2? Physics of Boundary tells us!!

If $C_1 > 0$, then $\mathcal{X}(x) = A e^{+\sqrt{C_1}\,x} + B e^{-\sqrt{C_1}\,x}$.

That's no good! Can't make that function vanish at $x=0$
(convince yourself!) and $x = a$!!

$C_1 = 0$ also no good, can't make $A + Bx$ vanish at $x = 0$
(Convince yourself!) and $x = a$

so $C_1 < 0$, call it $-k^2$ to make it obvious it's negative.
$+\ C_2 = -C_1 = +k^2$ is clearly positive.

so
$$\boxed{\begin{aligned} \mathcal{X}(x) &= A\sin(kx) + B\cos(kx) \\ \mathcal{Y}(y) &= C e^{+ky} + D e^{-ky} \end{aligned}}$$

Four undetermined coefficient, plus k !

To go further, we need to use more boundary conditions

3310 Notes 3-19

① $V(x=0)$ has to vanish (left wall is grounded)

so $X(0)=0$ which tells me $B=0$.

so $X(x) = A\sin(kx)$

② $V(y\to\infty)$ has to vanish (far away, Voltage must $\to 0$)

this tells me $C e^{ky} \to 0$ for large y, so $C=0$.

So we've got

$$V(x,y) = X(x)\,Y(y) = \underbrace{AD}\,\sin(kx)\,e^{-ky}$$

Looks like ~~who cares what you call this~~, it's just some constant!
need 2 constants, but really $\longrightarrow$ Call it C'

Third boundary condition:

③ $V(x=a) = 0$ has to vanish, right wall is grounded.

so $C'\sin(ka)\,e^{-ky} = 0$ for any y.

But I cannot now set $C'=0$, 'cause if I did, $V=0$,

+ that's not right at the bottom!

So we need $\sin(ka)=0$. This tells us k (the "separation constant") cannot be any old thing!!

$$k = \frac{n\pi}{a}.$$ So certain k's will work!

(n can be an integer, >0. But $n=0$ is bad, $n<0$ isn't different!)

3310 Notes 3-20

So far,

$$V(x,y) = C' e^{-ky} \sin(kx) \qquad \text{with} \quad k = \frac{n\pi}{a}$$

$(n = 1, 2, 3, \ldots)$

[n=0 gives you V=0, which is no good
n negative is the same sol'n, just changes sign of C']

This satisfies $\nabla^2 V = 0$ (by construction, but check if you like)

and all boundary conditions except $V(x, y=0) = V_0(x)$.

So can we use sep. of variables?

well, if $V(x,0) = C' \sin(kx) = V_0(x)$

then we're golden. So if we started with

$V_0(x) = C' \sin(kx)$, with $k = n\pi/a$, we're done!

we have V everywhere.

But gosh, we should be able to set $V_0(x)$ to be whatever we want, doesn't have to be $\sin\left(\frac{n\pi x}{a}\right)$ physically! Are we stuck? No. Because if we have $V_1(x,y)$ which "works" and another $V_2(x,y)$ which "works", then $aV_1 + bV_2$ also works. $(\nabla^2 aV_1 + bV_2 = a\nabla^2 V_1 + b\nabla^2 V_2 = 0 + 0 = 0)$

3310 Notes 3-21

we have many sol'ns. For any + integer n

$$V_n(x,y) = C_n \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}$$

satisfies $\nabla^2 V_n = 0$ (and 3 b.c.'s).

so $$V(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}.$$

also satisfies $\nabla^2 V = 0$ (and all 3 b.c.'s check!

this sum is still a sum of zeros at $x=0$, $x=a$, and $y=+\infty$)!

so Now the question is, can you pick your C_n's such that

$V(x,0) = V_0(x)$. If so, we've got it!!

this means $$\boxed{\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a} = V_0(x)}$$

(because $e^0 = 1$ when $y=0$)

remember, $V_0(x)$ is a given function, it's a b.c.

C_n is as yet undetermined. If we can pick C_n's, then we win.

But we can! This is Fourier's Trick, any (reasonable) function $V_0(x)$ (for $0<x<a$) can always be written, uniquely, like this!

3310 Notes 3-22.

Fourier's Trick (Math interlude).

The functions $\sin\frac{n\pi x}{a}$ are a complete, orthonormal set of fns.

$$\int_0^a \sin\frac{n\pi x}{a} \cdot \sin\frac{n'\pi x}{a}\,dx = \begin{cases} 0 & \text{if } n' \neq n \\ a/2 & \text{if } n' = n \end{cases}$$

check for yourself!

So if $\quad V_0(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a}$

then $\quad \sin\frac{n'\pi x}{a} \cdot V_0(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a} \sin\frac{n'\pi x}{a} \quad \leftarrow$ multiply both sides by $\sin\frac{n'\pi x}{a}$

$$\int_0^a V_0(x) \sin\frac{n'\pi x}{a}\,dx = \sum_{n=1}^{\infty} C_n \int_0^a \sin\frac{n\pi x}{a} \sin\frac{n'\pi x}{a}\,dx \quad \leftarrow \text{integrate both sides}$$

$$= C_{n'} \cdot \frac{a}{2}$$

all vanish except for $n = n'$

So $\quad \boxed{C_{n'} = \frac{2}{a} \int_0^a V_0(x) \sin\frac{n'\pi x}{a}\,dx}$ A formula for all the C_n's.

Once you have all the C_n's... you're done!

So this gives us $V(x,y)$ everywhere, no matter what $V_0(x)$ on the base we started with.

3310 Notes 3-23.

Example: $V_0(x) = V_0$, the base is <u>also</u> metal, just at high voltage.

$$C_{n'} = \frac{2}{a}\int_0^a V_0 \sin\frac{n'\pi x}{a}\,dx = \frac{2V_0}{a}\cdot\frac{-a}{n'\pi}\left(\cos\frac{n'\pi x}{a}\right)_0^a$$

$$= -\frac{2V_0}{\pi n'}\left(\cos n'\pi - 1\right)$$

Pick any/every n' ~~f~~rom $1-\infty$.

e.g $C_1 = -\frac{2V_0}{\pi}(-2) = 4V_0/\pi$

e.g $C_2 = -\frac{2V_0}{\pi\cdot 2}(1-1) = 0.$

In general $C_{n\,odd} = \frac{4V_0}{n\pi}$, $C_{n-even} = 0.$

thus $$V(x,y) = \sum_{n=1,3,5,\ldots}^{\infty} \underbrace{\frac{4V_0}{\pi}\cdot\frac{1}{n}}_{\text{this was } C_n}\cdot\sin\left(\frac{n\pi x}{a}\right)e^{-\frac{n\pi y}{a}}$$

ugly? But, analytic, calculable, ... we did it.

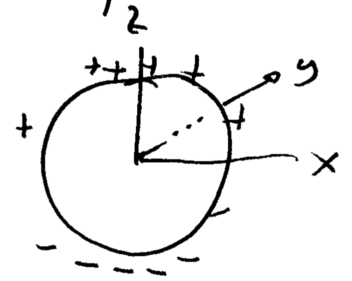
(<u>Any</u> function $V_0(x)$ can be used at the top, you'll get a different <u>set</u> of constants C_n's ...)

3310 Notes 3-24

- See Griffiths for more examples! Work through a couple...

Let's instead consider problems that have "boundaries" that look <u>spherical</u> instead of planar. So, $\mathcal{X}(x)\mathcal{Y}(y)\mathcal{Z}(z)$ will work, but it'll be a terrible mess. It's <u>much</u> more natural to try $\mathcal{R}(r)\Theta(\theta)\Phi(\varphi)$.

In <u>this</u> course, we will only look at problems with <u>no</u> φ dependence (e.g. a sphere of charge ρ could depend on θ (polar) but not φ.



$$\nabla^2 V = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + 0$$

Again, we <u>try</u> $V(r,\theta) = \mathcal{R}(r)\Theta(\theta)$, plug it in, and then <u>divide</u> both sides by $\mathcal{R}(r)\Theta(\theta)$, giving

$$\frac{\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\mathcal{R}(r)}{dr}\right)}{\mathcal{R}(r)} + \frac{\frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta(\theta)}{d\theta}\right)}{\Theta(\theta)} = 0$$

Multiply through by r^2, to get

3310 Notes 3-25

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2 R'(r)\right) + \frac{1}{\Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta\, \Theta'(\theta)\right) = 0$$

$$\underbrace{}_{C_1} + \underbrace{}_{C_2} = 0$$

$$C_1 + C_2 = 0$$

as before, no function of r alone can add to a fn of θ to give 0 unless each is a constant.

Once again, our PDE $\Rightarrow$ 2 simple ODE's.

" ", which needs to be positive, C_1 or C_2? This time we have a "built in" B.C, namely $\Theta(\theta)$ must not blow up (we can't allow infinite Voltages at finite r's! At least, not if we have smooth charge distributions) It turns out that this forces C_2 to have special features: it must be negative, and it must be writeable as $-\ell(\ell+1)$ with ℓ an integer (I won't prove this!)

So $C_1 = +\ell(\ell+1)$ for some integer $\ell \geq 0$
$C_2 = -\ell(\ell+1)$ (Here, $\ell = 0$ is not trivial!)

This ODE's are solvable. (Remember, you can just check!)

The general sol'n to $\frac{d}{dr}(r^2 R') = \ell(\ell+1) R$

is $R(r) = A r^\ell + B / r^{\ell+1}$

2 constants, to be determined by b.c.'s!

3310 Notes 3-26.

The angular eq'n is uglier. But the sol'ns are not so bad: For instance

$\ell=0$: $\frac{d}{d\theta}(\sin\theta\, \Theta'(\theta)) = 0$ is solved by $\Theta_0(\theta) = 1$

($\Theta(\theta)$ = constant works, + there is <u>another</u> function which, alas, blows up at $\theta=0$) I will <u>choose</u> $\Theta(0)=1$ and slide any constant coefficient over into my A and B of $R(r)$

$\ell=1$, the sol'n (which is finite) is $\Theta_1(\theta) = \cos\theta$

(check: $\frac{d}{d\theta} \sin\theta\, \Theta_1' = -(1)(2)(\Theta_1)\cdot\sin\theta$ it works!)

In general, $\Theta_\ell(\theta) = P_\ell(\cos\theta)$

with $P_\ell(\cos\theta)$ = "Legendre Polynomial"

$P_0(\cos\theta) = 1$

$P_1(\cos\theta) = \cos\theta$

$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$

Thus, $V_\ell(r,\theta) = R_\ell(r)\,\Theta_\ell(\theta) = \left(Ar^\ell + \frac{B}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

<u>solves</u> $\nabla^2 V_\ell = 0$. It's true for any ℓ, and so, like before, we can combine many V_ℓ's

3310 Notes 3-27

so $V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

satisfies $\nabla^2 V = 0$, (and $V(r,\theta)$ is finite for any theta .)

So here's the game: If I give you boundary conditions on a sphere,

like e.g. $V(a) =$ constant, or $V(a,\theta) = C * fn(\theta)$, or whatever,

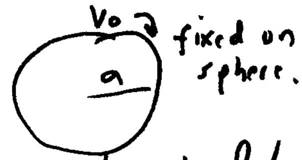
we can build a $V(r,\theta)$ which satisfies this b.c., and $\nabla^2 V = 0$,

by using the form above. You just need to find A_ℓ and B_ℓ's,

and uniqueness says if you find one such sol'n, that's it)

In the case $V(a,\theta) = V_0(\theta)$ is given, and suppose we

want $V(r,\theta)$ OUTSIDE this boundary



$\rightarrow$ we need $V(\infty,\theta) \rightarrow 0$, so A_ℓ better vanish $\forall \ell$!

so $V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$

unknown constants!

and $V(a,\theta) = V_0(\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{a^{\ell+1}} P_\ell(\cos\theta)$

Given

given constant

Known functions

This is again Fourier's trick!

3310 - Notes 3-28

Fourier's trick for Legendre polynomials:

I <u>claim</u> without proof.

$$\int_0^\pi P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta) \cdot \sin\theta\, d\theta = \begin{cases} 0 & \text{if } \ell \neq \ell' \\ \frac{2}{2\ell+1} & \text{if } \ell = \ell' \end{cases}$$

or, $x = \cos\theta$
$dx = -\sin\theta\, d\theta$

$$\int_{-1}^{1} P_\ell(x)\, P_{\ell'}(x)\, dx = \left.\begin{matrix} 0 \\ 2/_{2\ell+1} \end{matrix}\right\} \begin{matrix} \ell \neq \ell' \\ \ell = \ell' \end{matrix}$$

so to find B_ℓ, multiply both sides by $P_{\ell'}(\cos\theta)$ and integrate.

<u>All</u> terms in the sum $\sum_{\ell=0}^{\infty} \cdots P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta)$ will vanish but <u>one</u>

so $$\frac{B_{\ell'}}{a^{\ell'+1}} \cdot \frac{2}{2\ell'+1} = \int_0^\pi P_{\ell'}(\cos\theta) \cdot V_0(\theta) \sin\theta\, d\theta$$

This gives you all your B_ℓ's!

If I give you $V_0(\theta)$ on the sphere and want V <u>inside</u>,

the argument is similar, except now $B_\ell = 0$ (so $V(0)$ is finite)

so $$V_0(\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta),$$ with

and : $$A_{\ell'} \cdot a^{\ell'} \cdot \frac{2}{2\ell'+1} = \int_0^\pi P_{\ell'}(\cos\theta)\, V_0(\theta) \sin\theta\, d\theta$$

3310 - Notes 3-27

Comments: What we have is a method to find $V(r,\theta)$ if we know $V(a,\theta) \equiv V_0(\theta)$. The sol'n will be a sum, we need only to find some numerical coefficients (A_ℓ's if our region includes the origin, or B_ℓ's if our region includes $r=\infty$, assuming no funny business)

The integrals look scary, $\int_0^\pi P_\ell(\cos\theta)\, V_0(\theta) \sin\theta \, d\theta$.

In general, if $V_0(\theta)$ is nasty, we may be in trouble. The trick is to remember orthogonality, $\int_0^\pi P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta) \sin\theta \, d\theta = 0$ if $\ell \neq \ell'$

So this approach works best if $V_0(\theta)$ looks like a single, simple Legendre Polynomial (or perhaps, a sum of a couple)

e.g. if $V_0(\theta) = V_0$, we see that $V_0(\theta) = V_0 \, P_0(\cos\theta)$

and so we'll only need one integral, with $\ell = 0$, all others vanish.

$P_0 = 1$

$P_1 = \cos\theta$

$P_2 = \frac{3}{2}\cos^2\theta - \frac{1}{2}$

So [e.g. If $V_0(\theta) = \sin^2\theta$,

I notice this is $1 - \cos^2\theta$

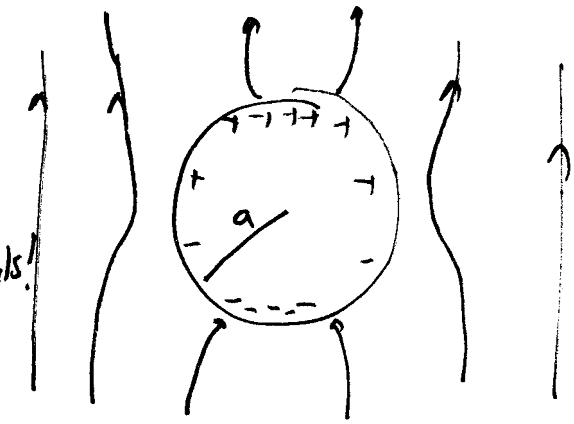
$= -\frac{2}{3} P_2 - \frac{1}{3} P_0$.

So only $\ell = 0$ and $\ell = 2$ terms will live!]

3310 3-30 Notes

<u>Classic Example</u> Let's put a metal sphere (radius a) into an existing, external, uniform field $\vec{E} = E_0 \hat{z}$. Now, this field will polarize the sphere, which in turn superposes, making a complicated field.

what is $V(r,\theta)$ everywhere?



• $V(r<a, \theta) = V_0$ conductors are equipotentials!

And I can pick one spot to be $V=0$, so let's let $V(0)=0$, which makes $V_0 = 0$.

Outside, well, we have new b.c.'s! $V(r=\infty, \theta)$ is not 0!

With $\vec{E}_{ext} = E_0 \hat{z}$, far away you must have $\vec{E}_{tot} \rightarrow E_0 \hat{z}$
$= -\vec{\nabla} V$

Thus $V(r\rightarrow\infty, \theta) \longrightarrow -E_0 z$ (no constant, because need $V(x\rightarrow\infty, z=0) = 0$ too!

So we have $V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$
$(r>a)$

Consider first $r\rightarrow\infty$, with $V(\infty,\theta) \rightarrow -E_0 z = -E_0 r\cos\theta$
$= -E_0 r P_1(\cos\theta)$

so $-E_0 r P_1(\cos\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell \cdot r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)\Big|_{r\rightarrow\infty} P_\ell(\cos\theta)$

3310 3-31 Notes

In, The right side, the B_ℓ terms don't contribute,

so $-E_0 r \, P_1(\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$

L.H.S. is pure P_1. So, only $\ell=1$ can contribute!!

so $A_1 r P_1 = -E_0 r P_1$ which means $A_1 = -E_0$ is fixed by

the B.C. at ∞

and so are all other A_ℓ's, all rest vanish!

Next, at $r=a$, we have

$$V(a,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{a^{\ell+1}} P_\ell(\cos\theta) + \underbrace{A_1 a \, P_1(\cos\theta)}_{\text{the only surviving } A_\ell \text{ term}}$$

$\parallel$

0

so how can this be zero? All B_ℓ's better vanish, except $\ell=1$,

which will have to kill that A_1 bit.

So $\frac{B_1}{a^{1+1}} = -A_1 a$. [so we have $\frac{B_1}{r^{1+1}} P_1(\cos\theta)$ term too]

thus $V(r,\theta) = A_1 r \, P_1(\cos\theta) + -\frac{A_1 a^3}{r^2} P_1(\cos\theta)$ with $A_1 = -E_0$

so $V(r,\theta) = E_0 P_1(\cos\theta)\left(\frac{a^3}{r^2} - r\right) = \underbrace{-E_0 r\cos\theta}_{\text{that was } E_{ext}!} + \underbrace{\frac{a^3}{r^2} E_0 \cos\theta}_{\text{the "induced" } V!}$

(and $= 0$ for $r \le a$)

- If, instead of giving you V at some radius, I gave you $\sigma(a,\theta)$, you can still use this approach, because

$$E^{\perp}_{out} - E^{\perp}_{in} = \frac{\sigma}{\varepsilon_0}$$ with a sphere, and $E^{\perp} = -\frac{\partial V}{\partial r}$,

$$\left.\frac{\partial V}{\partial r}\right|_{r=a+\varepsilon} - \left.\frac{\partial V}{\partial r}\right|_{r=a-\varepsilon} = -\sigma/\varepsilon_0 \qquad (B.C!)$$

So, you can use separation of variables, and separately treat

$r>a$ case (where all A_ℓ's vanish)

$r<a$ " (" " B_ℓ's vanish),

then use the (BC!) above to get a relation between A's + B's.

By the way, in previous Example, $V=0$ for $r \le a$

so $\partial V/\partial r|_{inside} = 0$

$$\left.\frac{\partial V}{\partial r}\right|_{just\ outside} = \frac{\partial}{\partial r}\left(-E_0 r\cos\theta + \frac{a^3}{r^2}E_0\cos\theta\right) = \left.-E_0\cos\theta - \frac{2a^3}{r^3}E_0\cos\theta\right|_{r=a}$$

so $\sigma = \varepsilon_0 E_0 \cos\theta\,(1+2) = 3\,\varepsilon_0 E_0 \cos\theta$

so our Polarized sphere has a simple $\sigma(\theta)$ on the surface, + at north pole, − at south pole, like you might expect.

(+ By subtracting the uniform field everywhere, you now know the V (and field) just from this σ alone)