Boundary Conditions

Electrostatics is about figuring out V (or E) from charges, or "setups" ← e.g. location + shapes of conductors + insulators.

The field is determined by what is at the boundaries and we often talk about "Boundary conditions" which establish $\vec{E}$ + V.

charges + Materials

ρ

$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho \, d\tau}{\mathcal{R}^2}\hat{\mathcal{R}}$ $\qquad$ $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$

$\nabla^2 V = \frac{-\rho}{\varepsilon_0}$

E the field

$\Delta V = -\int \vec{E}\cdot d\vec{L}$ $\qquad$ $\vec{E} = -\vec{\nabla} V$

V

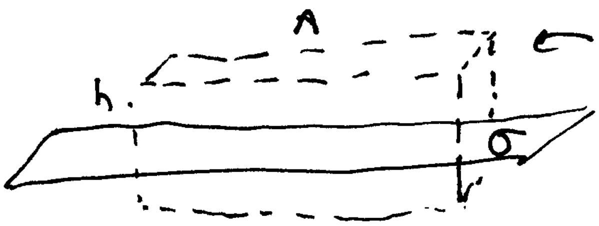
$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{\mathcal{R}} d\tau$

the potential

At physical boundaries: (e.g. sheet w. charges)

- $\vec{E}$ can "jump" (because charges create field)
- but $\boxed{V \text{ is always continuous!}}$ ← Nice!!

How does $\vec{E}$ jump? By Gauss' Law.



← Small "pillbox"

Let $A \to 0$

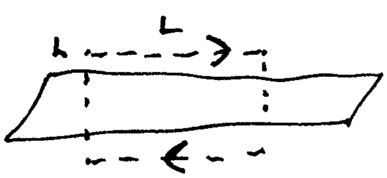
$h \to 0$

$$\int \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma \cdot A}{\varepsilon_0}$$

$$\vec{E} \cdot \hat{n}_{above} A - \vec{E} \cdot \hat{n}_{below} A = \frac{\sigma A}{\varepsilon_0}$$

$$\boxed{\text{so } \underline{\text{change}} \text{ in } \vec{E} \cdot \hat{n} \text{ is } \sigma/\varepsilon_0}$$

How about $\vec{E}$ parallel to sheet? Use $\vec{\nabla} \times \vec{E} = 0$



on Loop, $E_{\parallel}^{above} \cdot L - E_{\parallel}^{below} \cdot L = 0$

if $h \to 0$

so $\boxed{E_{\parallel} \text{ to sheet is } \underline{\text{continuous}}}$

Since $\vec{E} = -\vec{\nabla} V$ can rewrite these expressions in terms of V. We'll return to this!

$\boxed{\text{ENERGY}}$: Recall, we invented $V(\vec{r})$ = "Voltage"

$$V(\vec{r}) = -\int_{\substack{\text{origin where}\\ V=0}}^{\vec{r}} \vec{E}\cdot d\vec{l} \qquad \leftarrow \text{So, given } \vec{E}\text{, can always compute } V.$$

then we showed, mathematically & with Maxwell's help

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{vol} \frac{\rho(r')\,d\tau'}{\mathscr{R}} \qquad \leftarrow \text{So, given } \rho\text{, can always compute } V$$

$$\vec{E} = -\vec{\nabla} V \qquad \leftarrow \text{so, once have } V\text{, you know } \vec{E}!$$

$$\nabla^2 V = -\rho/\varepsilon_0 \qquad \leftarrow \text{So, " " ", " " } \rho!$$

But what is it? What does it mean?

Think of moving a tiny test charge q through E fields from a to b. $\vec{F}_{electric} = q\vec{E}$, so $\vec{F}_{\underline{you}} = -q\vec{E}$ to "fight the field"

To go from a to b, you do $W_{\substack{ext\\ a\to b}} = +\int_a^b \vec{F}_{you}\cdot d\vec{l}$

$$= -q\int_a^b \vec{E}\cdot d\vec{l}$$

Look up top,

$$\boxed{W_{\substack{ext\\ a\to b}} = q\,\Delta V_{ab} = q\big(V(b) - V(a)\big)}$$

So Voltage carries info about work/energy!

In 1110, if <u>you</u> <u>do</u> work, can talk about stored potential energy. Here, $\boxed{PE \equiv qV}$ (from prev. page)

(Note, always ambiguity, can <u>define</u> PE = 0 anywhere)

so $V(\vec{r}) = PE/q$ = $\dfrac{\text{the potential energy}}{\text{unit charge}}$ (= + Work by <u>you</u> to bring q from where V = 0, to point $\vec{r}$)

[Could call the p.e = $U(\vec{r}) = qV(\vec{r})$
Griffiths calls it W, it's work needed by <u>you</u> to get q to this point.]

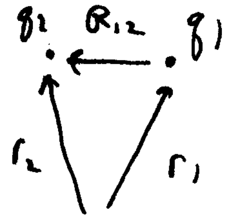
- That's "PE of a charge q in presence of <u>others</u>". But what about work to get <u>others</u> together?! Start from scratch, <u>build</u> up <u>any</u> given distribution of q's, how much work? That will tell us "stored electrostatic energy of system"

So let's bring in, one at a time, $q_1, q_2, q_3 \ldots$ & figure out total work.

3310 (1-2) - 38

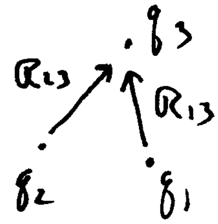
Bring in q_1. No other q's $\Rightarrow$ no work.

Bring in q_2. q_1 is there, producing field.



so $W_2 = q_2 \cdot V_{\substack{\text{caused}\\ \text{by } q_1}} = q_2 \cdot \left(\frac{1}{4\pi\varepsilon_0} \frac{q_1}{\mathcal{R}_{12}}\right)$

Now bring in q_3.



$W_3 = q_3 \; V_{\substack{\text{caused by}\\ 1 \text{ and } 2}} = q_3 \cdot \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{\mathcal{R}_{13}} + \frac{q_2}{\mathcal{R}_{23}}\right)$

Total so far is $W_1 + W_2 + W_3$

$$W_{system} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{\mathcal{R}_{12}} + \frac{q_1 q_3}{\mathcal{R}_{13}} + \frac{q_2 q_3}{\mathcal{R}_{23}}\right)$$

see pattern? Extends to any #.

Add all pairs $\frac{q_i q_j}{\mathcal{R}_{ij}}$
- but don't compute "self energy" $i=j$
- and don't double count,

$\left(\frac{q_j q_i}{\mathcal{R}_{ji}} \text{ is same thing}\right)$

or, do double count and then divide by 2!!

so $W_{system} = \frac{1}{2} \cdot \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} q_i q_j / \mathcal{R}_{ij}$

Note: Could be negative!

Can reorganize this

$$W_{sys} = \frac{1}{2} \sum_{i=1}^{n} q_i \left(\underbrace{\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{q_j}{4\pi\epsilon_0 R_{ij}}} \right)$$

what's in parens? It looks like $\tilde{V}(\vec{r}_i)$

the voltage you get at point "i" due to all the other charges at all points $j \neq i$. (Being careful not to include "self energies")

so $W_{sys} = \frac{1}{2} \sum_{i=1}^{n} q_i \, \tilde{V}_i(\vec{r}_i)$ (No "bringing in q's one at a time" any more)

remember, needed this because we otherwise would double count energy 1-2 and 2-1.

This is a nice expression because pretty clear what to do now if charges "smear", i.e. $\rho(r)$ instead of q_i.

$$W_{sys} = \frac{1}{2} \int dq \cdot \tilde{V}(\vec{r}) = \frac{1}{2} \int \tilde{V}(\vec{r}') \rho(r') d\tau'$$

[Here $\tilde{V}(\vec{r})$ = the potential at point $\vec{r}$ due to all of ρ except right at $\vec{r}$, but this is irrelevant issue for ρ (!) there is no charge in an infinitesimal volume...]

3310 (1-2) -40.

$W_{sys} = \frac{1}{2}\int \rho(r)\, V(r)\, d\tau$ ← Total Energy of a system.

where is it located? In the E fields! Let's see how...

$\rho = +\varepsilon_0 \vec{\nabla}\cdot\vec{E}$, so $W_{sys} = \frac{+\varepsilon_0}{2}\int_{vol} (\vec{\nabla}\cdot\vec{E})\, V\, d\tau$

But $\int u\, dv = uv| - \int v\, du$, or in 3-D see eq'n 1.59

$$W_{sys} = +\frac{\varepsilon_0}{2}\left[\int_{\text{Boundary}} V\vec{E}\cdot d\vec{A} - \int_{\text{Vol}} \vec{E}\cdot\vec{\nabla}V\, d\tau\right]$$

If Volume is "all space", $V, \vec{E} \to 0$ far away!

So, as long as all charges are localized (no good for "∞ sheet", e.g!)

$W_{sys} = -\frac{\varepsilon_0}{2}\int_{vol} \vec{E}\cdot\vec{\nabla}V\, d\tau$ But $\vec{\nabla}V = -\vec{E}$, so

$$\boxed{W_{syst} = +\frac{1}{2}\varepsilon_0 \int_{vol} E^2\, d\tau}$$

Cool. It's E that stores the energy,

$\frac{1}{2}\varepsilon_0 E^2$ tells you "stored energy / m^3".

[But, should really only use this with continuous ρ's.
If have discrete point charges, go back to the sum formula.]

Conductors : Perfect conductance is idealization (though realizable w. superconductors) but metals (Cu, Al, etc) are excellent approximations!!

→ Charges are free inside : can respond (instantly + w/o loss) to forces. Consequences in e-static situations (!!)

- $\vec{E} = 0$ inside . See Griffiths p. 97, but clearly if $\vec{E} \neq 0$, $\vec{F} = q\vec{E} \neq 0$, charges will move! Keep moving until $\vec{E} = 0$. (that's "static", then)

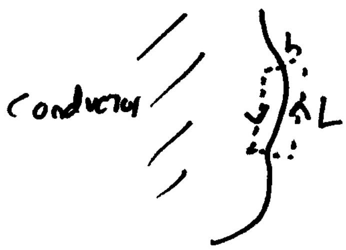
- $\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E} = 0$ inside

- Excess charge must live on outside edge. See !

- $\Delta V(a \to b) = -\int_a^b \vec{E} \cdot d\vec{L} = 0$ if a, b are both in or on conductor. So ⇒ Equipotential throughout. (Even if charged!)

- $\vec{E} \perp$ surface at edges. (If there was any $E_{\parallel}$, then surface charges would move) (or, see next page)

3310 (1-2) - 42

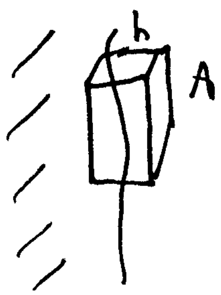


Consider $\oint \vec{E} \cdot d\vec{L}$, with $h \to 0$. Must be 0!

so get 0_{inside} + tiny leg + $E_{\parallel} \cdot L$ + tiny leg $= 0$.

This is a formal proof that $E_{\parallel}$ (outside) $= 0$.

Also, consider Gauss pillbox, with <u>tiny</u> height h.



$E_{out} \cdot A + E_{in} \cdot A$ + tiny walls $= \dfrac{Q_{enc}}{\varepsilon_0} = \dfrac{\sigma \cdot A}{\varepsilon_0}$

so $E_{out} = \dfrac{\sigma}{\varepsilon_0}$, pointing <u>normal</u>.

**

Interesting, it's <u>not</u> $\dfrac{\sigma}{2\varepsilon_0}$ like an isolated sheet of charge gives.

(there <u>must</u> be other charges superposing to <u>give</u> us this E field)

Many many consequences!

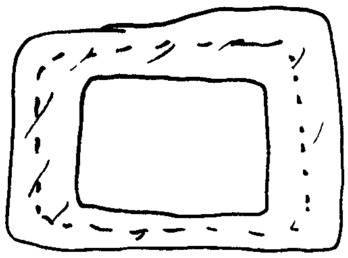
- Conductors <u>polarize</u> in presence of external q's.

  <u>Have</u> to, to ensure $\vec{E} = 0$ inside.

- makes it harder to <u>solve</u> for $V(r)$ (or $\vec{E}$), since no longer know "a-priori" ρ, it will ad<u>just</u> itself!

(So e.g. $V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho}{\mathscr{R}} d\tau$ is still true... but what is ρ now?)

Cavities & "shielding".



If conductor has hole, what's going to happen? $\vec{E}=0$ in metal region, but what about in hole?

$\cdot Q_{ext}$

Gauss for dashed line says $\iint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$

But $\vec{E}=0$ on dashed line! So $Q_{enc}=0$.

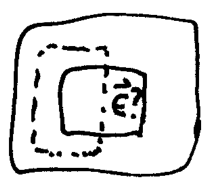
- If there is a q in the hole, this says $-q$ appears at "inside edge"
- If there is no q in the hole, I claim $\vec{E}=0$ in hole too!

why?

Arg #1: Suppose you start with solid conductor. We know $E=0$ + $\rho=0$ throughout. Now remove hole material.

Not removing any Q's so not changing $\vec{E}$ anywhere! (It's Q's that create $\vec{E}$'s, after all)

Arg #2: If $\vec{E}\neq 0$, then



this loop shows $\int \vec{E}\cdot d\vec{l} = 0$

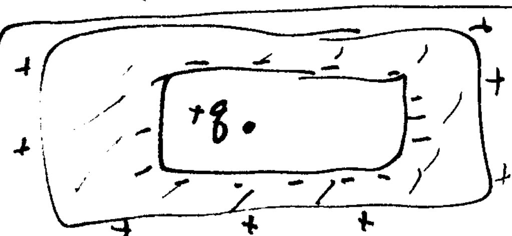
but it's 0 for any line, in either direction! No way (unless $\vec{E}=0$ everywhere.)

Arg #3: We'll soon learn "uniqueness theorem":

Once you find a sol'n for V (or E) throughout space that is consistent with "boundary conditions", there is no other sol'n.

$\vec{E}=0$ is consistent, + it's unique, so that's what it is.

This is "Faraday cage" effect: Inside a conductor, $\vec{E}=0$, (even in cavities, even with Q's outside, even if conductor has q.)



· what if put q in there?

- We know you polarize, (putting $-q$ on inside wall...)

, Charge conservation puts $+q$ on outside wall.

$E \neq 0$ inside the cavity any more!

· outside: field $\neq 0$ (because $Q_{enc}=+q$)

Field outside is same as if had same conductor with no cavity, but net charge $+q$. (Again, that "uniqueness theorem")

3310 (1-2)-45

Forces : Consider a sheet (conductor or not) with surface charge σ. Apply an $\vec{E}$ ~~external~~ field ... what force would a patch (area dA) feel?

well, $d\vec{F} = dq\,\vec{E} = (\sigma \cdot dA)\,\vec{E}$. But what is $\vec{E}$? If you're on a surface, $\vec{E}$ is not continuous, $E_{above} - E_{below} = \frac{\sigma}{\varepsilon_0}$

Answer : use $\vec{E}$ from all q's except the patch, 'cause nothing exerts a force on itself!

Superposition: $\vec{E}_{total} = \vec{E}_{external} + \vec{E}_{\substack{patch\\ itself}}$

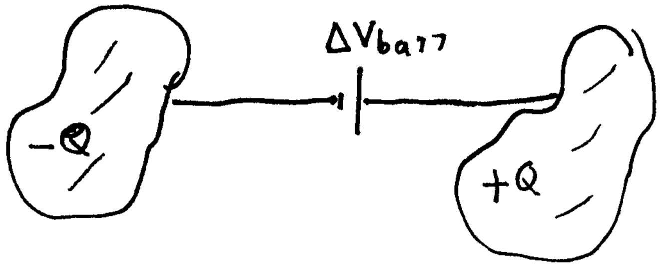
$\vec{E}_{external}$ will be continuous (!) so
$$\vec{E}_{above} = \vec{E}_{ext} + \frac{\sigma}{2\varepsilon_0}$$
$$\vec{E}_{below} = \vec{E}_{ext} - \sigma/2\varepsilon_0$$

thus $\vec{E}_{ext} = \frac{1}{2}(\vec{E}_{above} + \vec{E}_{below})$. Use the average real fields.

For conductor, $\left.\begin{aligned}\vec{E}_{above} &= \sigma/\varepsilon_0 \\ \vec{E}_{below} &= 0\end{aligned}\right\} \Rightarrow d\vec{F} = \sigma\, dA \cdot \frac{\sigma}{\underline{\underline{2}}\varepsilon_0}$ (out)

thus, there is an outward pressure $\frac{d\vec{F}}{dA} = \frac{\sigma^2}{2\varepsilon_0}$ outward.

CAPACITORS: Any pair of conductors will have a well defined ΔV ('cause each one is an equipotential)



← you can even choose it, like this!

there will be $\vec{E}$ fields around/between them, so $\Delta V = -\int_a^b \vec{E} \cdot d\vec{L}$

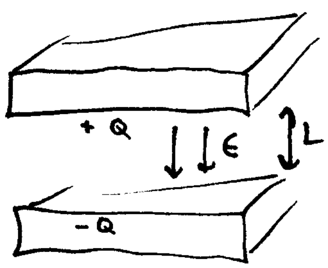
But Gauss' law says $E \propto Q$, so $\Delta V \propto Q$, so

Define $C = \dfrac{Q}{\Delta V}$. [Depends on objects, + config, + shape, but not on Q or ΔV!]

1 C/volt = 1 Farad.

Some situations we easy to calculate, if know $\vec{E}$ field

Ex 1:



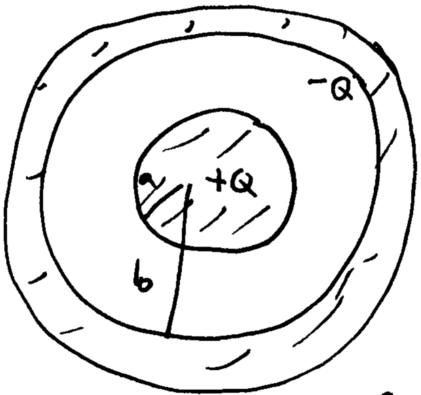
Large, parallel plates.

$\vec{E} = \dfrac{\sigma}{\varepsilon_0}$ between, 0 elsewhere

$\Delta V = -\int \vec{E} \cdot d\vec{L} = +\dfrac{\sigma}{\varepsilon_0} \cdot L$ ← (Do you see why it's + ?)

so $C = \dfrac{Q}{\Delta V} = \dfrac{Q}{Q/A \; L/\varepsilon_0} = \dfrac{\varepsilon_0 A}{L}$ //

Ex 2:



Here $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$

$$\Delta V = -\int_a^b \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0} Q \cdot \left.-\frac{1}{r}\right|_a^b$$

and $C = \frac{Q}{\Delta V} = 4\pi\varepsilon_0 \left(\frac{1}{\frac{1}{a}-\frac{1}{b}}\right)$

[If $b \to \infty$, this is $4\pi\varepsilon_0 a$]

<u>Energy stored</u>: Could compute $\frac{\varepsilon_0}{2}\int E^2 d\tau$,

but can also ask "how much external work needed" to charge it up to Q?

Each dq that gets moved takes work

$dW_{\text{to move } dq \text{ over}} = dq \cdot \Delta V$, [but $\Delta V = Q/C$ depends on Q we've built up so far.]

$$W_{tot} = \int_{q=0}^{q=Q} \Delta V \cdot dq = \int_0^Q \frac{q}{C}\, dq = \frac{1}{2}\frac{Q^2}{C}$$

(So, our sphere above has stored energy $\frac{1}{2}\frac{Q^2}{(4\pi\varepsilon_0 a)}$ =