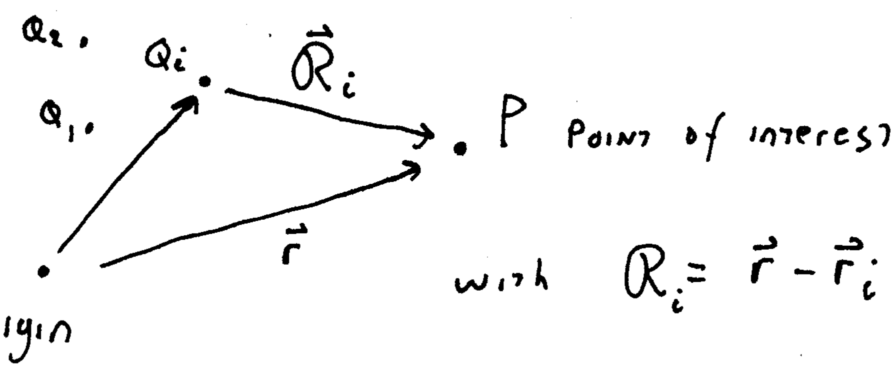
$\vec{E}$ field : $\vec{E} = \dfrac{\vec{F}_{\text{on a test } q}}{q}$ (Let $q \to 0$, so doesn't mess w. other q's positions)

From last time, this means

$$\underset{\text{due to } Q_i\text{'s}}{\vec{E}_{\text{at } P}} = \sum_{i=1}^{N} \frac{Q_i}{4\pi\varepsilon_0} \frac{\hat{R}_i}{R_i^2}$$

Reminder:



with $\vec{R}_i = \vec{r} - \vec{r}_i$

- $\vec{E}$ fields are vectors
- Present in empty space
- Calculable (easily) if know where all other Q_i's are.
- Tells you $\underset{\substack{\text{small } q \\ \text{at } P}}{\vec{F}_{\text{on any}}} = q\,\vec{E}_{\text{at } P}$.

Continuous charges: what if Q_i's are "smeared"?

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{Q_i}{R_i^2}\hat{R}_i \longrightarrow \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{R^2}\hat{R}$$

Here $\hat{R}$ points FROM source (dq) TO point labeled $\vec{r}$.

(Subscript i is now hidden, be sure you can visualize this)

In 3-D, Volume charge density $\rho(\vec{r}) \equiv \dfrac{\text{charge}}{\text{volume}}$

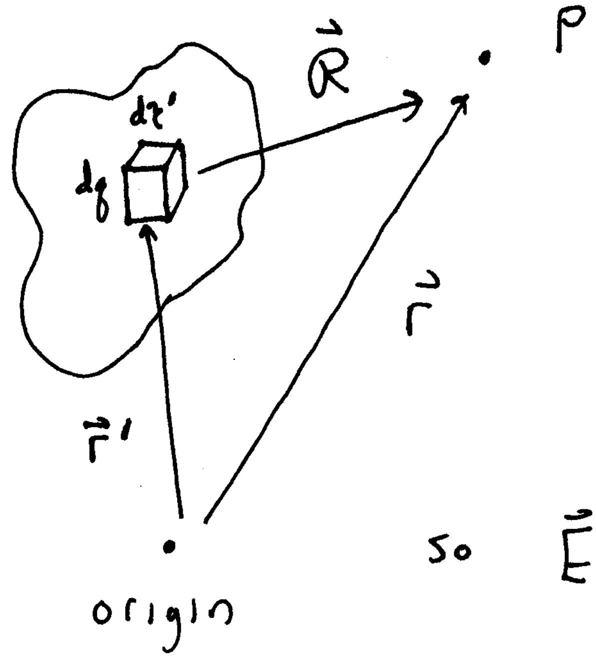
IDEALIZATION!

$$dq = \rho \cdot d(\text{Volume})$$ do you see this?

Griffiths calls $d(\text{Volume}) = d\tau'$

might also call it $d^3\vec{r}' = dx'\,dy'\,dz'$.

why the prime? Draw a picture!



As usual, $\vec{R} = \vec{r} - \vec{r}\,'$.

Summing over dq's means

$$\int \rho(\vec{r}\,')\,d^3\vec{r}\,' \ldots$$

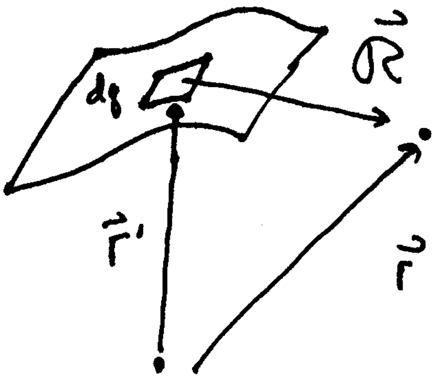
so $$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_{\text{volume}} \frac{\rho(\vec{r}\,')\,d^3\vec{r}\,'}{R^2}\hat{R}$$

with $\vec{R} = \vec{r} - \vec{r}\,'$.

If charges live on flat (or 2-D) surface, use

$\sigma(\vec{r}) = \dfrac{\text{charge}}{\text{area}}$ (near $\vec{r}$)

so $dq = \sigma \cdot d(\text{Area})$



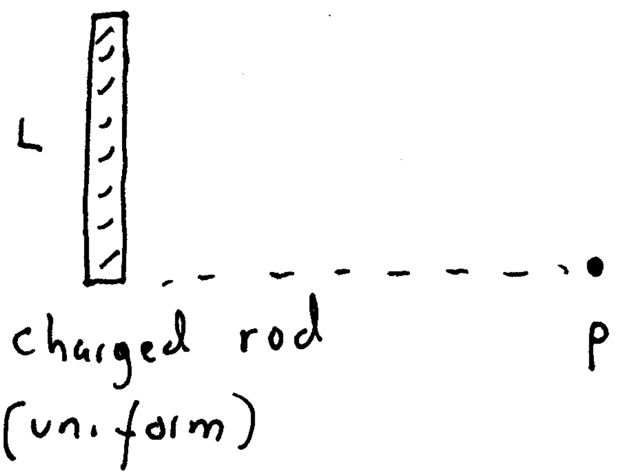
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iint_{\text{Area}} \frac{\sigma(\vec{r}')\, d^2(\text{Area})}{R^2} \cdot \hat{R}$$

If charges all live on a line (1-D), use

$\lambda = \dfrac{\text{charge}}{\text{length}}$

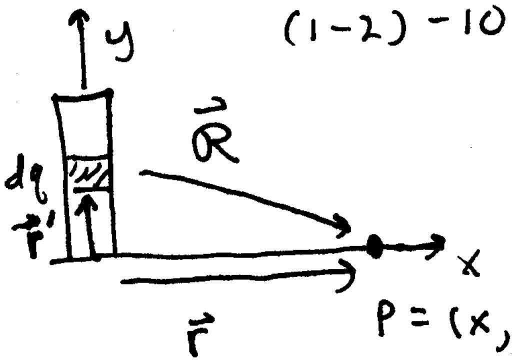
Draw your own picture, what's $\vec{E}$?

Example: Like Griffiths 2.1, but with a twist



Let's find $\vec{E}$ at the point shown.

[Try it yourself, first]



$\lambda = Q/L$ is given.

Find $\vec{E}(x,0,0)$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}')\,d\ell'}{R^2}\,\hat{R}$$

That's the formula. But, I like to go back to basics - look at the picture + think about what E should be

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{R^2}\,\hat{R}$$

- Dang, no symmetry! I'll find Ex, you try Ey if you like... *
- $dq = \lambda\, dy$ (Do you see it? Look @ the picture! Here, $d\ell'$ in the integral IS just dy)
- What's $\vec{R}$? Just look!

$\vec{r} = (x, 0, 0)$

$\vec{r}\,' = (0, y, 0)$ ← Do you see that?

$$R = \vec{r} - \vec{r}\,' = (x, -y, 0)$$

To get $\vec{E}_x$, just use $\hat{R}_x = \dfrac{\vec{R}_x}{|R|}$

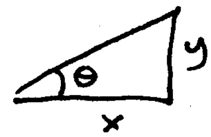
$R = \sqrt{x^2 + (-y)^2}$, $\vec{R}_x = x$, $\vec{R}_y = -y$, $\hat{R}_x = \dfrac{x}{\sqrt{x^2+y^2}}$

so $$E_x(x,0) = \frac{1}{4\pi\varepsilon_0}\int_{y=0}^{y=L} \frac{\lambda\, dy}{x^2+y^2}\cdot\frac{x}{\sqrt{x^2+y^2}}$$

What do you do when faced w. such an integral?

I would ① Look it up
② Use Mathematica or
③ Do it!

Let $y = x \tan\theta$



so $dy = x \cdot \frac{1}{\cos^2\theta} \cdot d\theta$

$$(x^2+y^2)^{3/2} = x^3\left(1+y^2/x^2\right)^{3/2} = x^3\left(1+\tan^2\theta\right)^{3/2} = x^3\left(\frac{1}{\cos^2\theta}\right)^{3/2}$$

so we have $\frac{\lambda}{4\pi\varepsilon_0}\int \frac{(x/\cos^2\theta)\,d\theta}{x^3\cdot(1/\cos^3\theta)}\cdot x = \frac{\lambda}{4\pi\varepsilon_0 x}\cdot\int \cos\theta\, d\theta$

$$\vec{E}(x,0) = \frac{\lambda}{4\pi\varepsilon_0 x}\cdot \sin\theta\Big| = \frac{\lambda}{4\pi\varepsilon_0 x}\,\frac{y}{\sqrt{x^2+y^2}}\Bigg|_{y=0}^{L} = \frac{\lambda L}{4\pi\varepsilon_0 x\sqrt{x^2+L^2}}$$

Look at picture, $\sin\theta = y/\sqrt{x^2+y^2}$

(Griffiths' Example went $-L$ to L!)

Does it make <u>sense</u>?

Units: λ is Coul/meter, so $\frac{1}{4\pi\varepsilon_0}\frac{[C/m \cdot m]}{m\cdot m}$ Good!!

Sign: is +, makes good physical sense.

Limits: often very good to look at →

$\vec{E}(x,0)$ if $x \gg L$, the "rod" looks small, like a point...

Let's see what the math limit gives:

$$\frac{1}{\sqrt{x^2+L^2}} = \frac{1}{\sqrt{x^2(1+L^2/x^2)}} = \frac{1}{x\sqrt{1+\varepsilon}} \quad \text{with } \varepsilon = \frac{L^2}{x^2} \ll 1$$

$$(1+\varepsilon)^n = 1 + n\varepsilon + \frac{n(n-1)\varepsilon^2}{2!}\dots \quad \text{so} \quad (1+\varepsilon)^{-1/2} = 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 + \dots$$

$$\vec{E}(x \gg L, 0) = \frac{\lambda L}{4\pi\varepsilon_0 x} \cdot \frac{1}{x}\left(1 - \frac{1}{2}\frac{L^2}{x^2} + \dots\right)$$

$$= \underbrace{\frac{\lambda L}{4\pi\varepsilon_0 x^2}}\left(1 - \underbrace{\frac{1}{2}L^2/x^2} + \dots\right)$$

Cool! That's $\frac{Q}{4\pi\varepsilon_0 x^2}$

I like that this is less than 1, does that make sense to you? The field is a little less than a point @ the origin would give...

(c7)

If $x \to 0$, we're snug up to charges.... $\frac{1}{\sqrt{x^2+L^2}} \simeq \frac{1}{L}(1+\dots)$

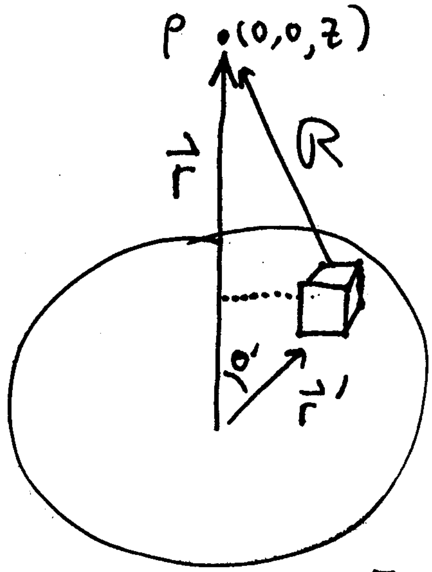
$$\vec{E}(x \ll L, 0) = \frac{\lambda L}{4\pi\varepsilon_0 x \cdot L}$$ this blows up,

(although like $1/x$, not $1/x^2$)

Another example. Sphere, uniform ρ, find $\vec{E}$ outside.

You could put P anywhere, but by symmetry, all pts are equiv (if same radius away). So, pick easy one.

If $\vec{r} = (0, 0, z)$, $E_x = E_y = 0$ by symmetry (convince youself!)



$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{R^2} \hat{R}$$

here: $dq = \rho \, d\tau' = \rho \, dx\,dy\,dz$

$$= \rho \, r'^2 dr' \sin\theta' d\theta' d\phi'$$

To get $\vec{R}$, look at picture!

$\vec{r} = z\hat{k}$, $\vec{r}' =$ shown

$$|R| = \sqrt{z^2 + r'^2 - 2zr'\cos\theta'} \quad \leftarrow \text{Law of cosines!}$$

$$\hat{R}_z = \frac{\vec{R}_z}{|R|} = \frac{z - r'\cos\theta'}{|R|} \quad \leftarrow \text{Look at picture, convince yourself}$$

this integral is ugly but doable...

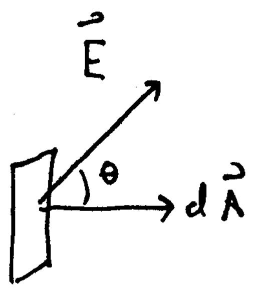
But wait, there is a better way!

Gauss' Law

$$\oiint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

- Exp'tal, a law of nature.
- Can "derive" from Coulomb's law in electrostatics *

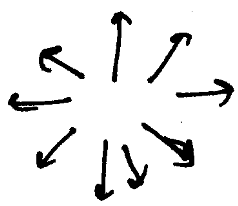
This is flux



$$\vec{E}\cdot d\vec{A} = E\, dA \cos\theta = E_\perp \cdot \text{Area}$$

That's the idea of flux: Proportional to Area + the amount of E "poking through".

If draw field lines



then $|E|$ is represented by density of lines

(in 3-D! 2-D pictures can fool you)

so $\frac{\#\text{ of lines}}{\text{area}} \leftrightarrow |E|$, which means flux is represented by how many lines "poke" through area.

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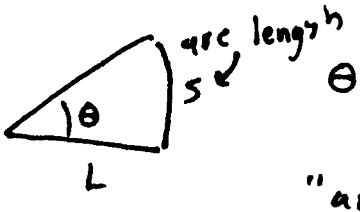
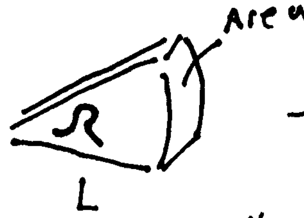
* (Griffiths shows that Gauss' law follows from Coulomb's law, if the surface is a sphere centered around one q.)

Can we show Gauss' law for ugly surface, q off center?



$\theta \equiv \dfrac{s}{L}$

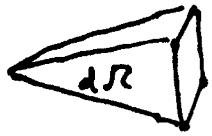
"angle"



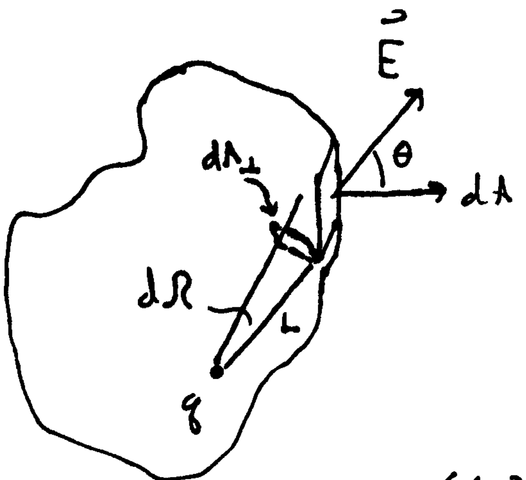
$\Omega \equiv \dfrac{\text{Area}}{L^2}$

"solid angle"

For small Areas,



$d\Omega = \dfrac{\text{Perpendicular Area}}{L^2} = \dfrac{dA_\perp}{L^2}$



$$\vec{E} \cdot d\vec{A} = \frac{q}{4\pi\varepsilon_0} \frac{\cos\theta \cdot dA}{L^2}$$

But note $d\Omega = \dfrac{dA_\perp}{L^2} = \dfrac{dA \cdot \cos\theta}{L^2}$

so $$\oiint \vec{E} \cdot dA = \frac{q}{4\pi\varepsilon_0} \iint d\Omega = \frac{q}{\varepsilon_0} \; !$$

If add more q's, by superposition

$$\oiint \vec{E} \cdot dA = \frac{q_{tot,\text{enclosed}}}{\varepsilon_0}.$$

Figure out for yourself why q's outside do <u>not</u> contribute!

Some vector calculus (at last!) [see Griff 1.56]

Divergence theorem (= Gauss' theorem, or Green's theorem)

$$\underset{\text{volume}}{\iiint} \vec{\nabla}\cdot\vec{F}\, d\tau = \underset{\text{closed } S}{\oiint} \vec{F}\cdot d\vec{A} \qquad \text{for } \underline{\text{any}} \text{ function } \vec{F}.$$

this is a "3-D version" of $\underbrace{\int_a^b \frac{df}{dx}\, dx}_{\text{integral of a deriv}} = \underbrace{f(b) - f(a)}_{\text{function on boundary}}$

<u>Meaning</u>: If $\vec{F}$ is any kind of "flow", then

$\vec{F}\cdot d\vec{A}$ = flux exiting dA

$\vec{\nabla}\cdot\vec{F}$ = "divergence", it's the spread from a point, or the "creation" of arrows

so $\iiint \vec{\nabla}\cdot\vec{F}\, d\tau$ = total spread created at <u>all</u> points

$\oiint \vec{F}\cdot d\vec{A}$ = total outflow.

what goes out must have originated from <u>sources</u> inside

Always true, so <u>also</u> for $\vec{E}$ fields.

$$\oint\!\!\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\iiint \rho \cdot d\tau \quad \leftarrow \text{Physics!}$$

$$\oint\!\!\oint \vec{E}\cdot d\vec{A} = \iiint \vec{\nabla}\cdot\vec{E}\, d\tau \quad \leftarrow \text{Math!}$$

True for any and all volumes, so integrands must agree

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}} \quad \text{at all points.}$$

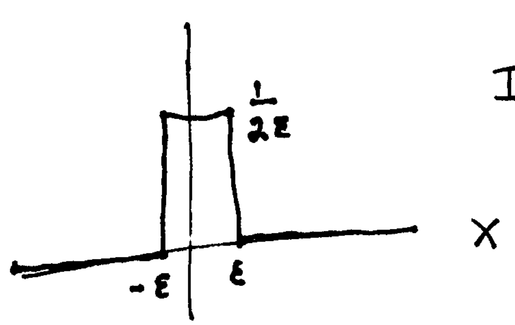
Suppose you have a point charge, so $\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$

- what should $\rho(\vec{r})$ look like??
- what's $\vec{\nabla}\cdot\vec{E}$?

Need a Math INTERLUDE: δ functions.

In 1-D, $\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$ such that $\int_{-\text{anything}}^{+\text{anything}} \delta(x)\,dx = 1$

It's a limit of sensible functions, like



It's "Tall + skinny", with area 1.

Let it get very skinny, that's $\delta(x)$

Note: $\displaystyle\int_{-\text{anything}}^{+\text{anything}} f(x)\,\delta(x)\,dx = f(0)$ ← (this defines the δ fn, actually)

In 3-D, $\iiint \delta^{(3)}(\vec{r})\, f(\vec{r})\, d\tau = f(0)$.

(Same deal, really!)

So a point charge has
- No charge density away from origin
- ∞ " " at origin

But $Q = \iiint \rho \cdot d\tau$ is finite.

$\rho(\vec{r}) = Q\,\underbrace{\delta^{(3)}(\vec{r})}$ is exactly that.

very "concentrated" at one point, but finite integral.

Side Note: $\displaystyle\int_{-\infty}^{\infty} f(x)\,\delta(ax+b)\,dx$ Let $u = ax+b$, $du = a\,dx$

$\displaystyle = \int_{-\infty}^{\infty} f\left(\frac{u-b}{a}\right)\delta(u)\cdot\frac{du}{a} = \frac{1}{a}\cdot f$ (the x for which the δ-fn argument vanished, namely $-b/a$)

Except if $a<0$, then integration limits are flipped, + you get $-\frac{1}{a} f(-b/a)$

i.e. $\delta(ax) = \frac{1}{|a|}\delta(x)$.

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Back to $\vec{\nabla}\cdot \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$. focus on $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)$

Method 1: Front flyleaf! $\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)$

Here, $\underline{\underline{v_r = 1/r^2}}$, so $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{1}{r^2}\right) = 0$.

Method 2: $\vec{\nabla}\cdot\left(\frac{1}{r^3}\vec{r}\right) = \left(\nabla\frac{1}{r^3}\right)\cdot\vec{r} + \frac{1}{r^3}\vec{\nabla}\cdot\vec{r}$

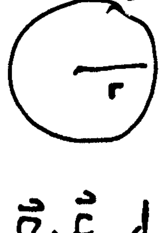
$$\left(\frac{-3}{r^4}\hat{r}\right)\cdot\vec{r} + \frac{1}{r^3}(3) = 0$$

Looks like $\vec{\nabla}\cdot\vec{E} = 0$ for point charge.

It $\underline{\underline{\text{is}}}$, except maybe at $r=0$, where all those formulas above involve r^2/r^2, not well defined.

So look again at Gauss: $\int \vec{\nabla}\cdot\vec{E}\,d\tau = \oiint \vec{E}\cdot d\vec{A}$

Imagine a $\underline{\underline{\text{tiny}}}$ surface. around origin $\quad$ RHS $= \iint \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}\cdot d\vec{A}$



$$= \frac{q}{4\pi\varepsilon_0 r^2}\int dA = \frac{q}{\varepsilon_0}$$

So $\int \vec{\nabla}\cdot\vec{E}\,d\tau = q/\varepsilon_0$ No matter what.

$\vec{\nabla}\cdot\vec{E}$ must be a δ fn! $\quad \vec{\nabla}\cdot\vec{E} = \frac{q}{\varepsilon_0}\cdot\delta^{(3)}(\vec{r})$

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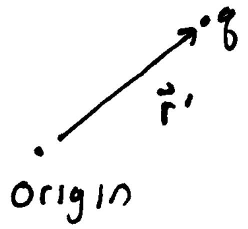
So $\rho(\vec{r}) = q\,\delta^{(3)}(\vec{r})$

And we've just learned $\vec{\nabla}\cdot \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} = \frac{q}{\varepsilon_0}\,\delta^{(3)}(\vec{r})$

So $\underline{\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 4\pi\,\delta^{(3)}(\vec{r})}$ // ← (see Griff. 1.100)

Summary: Point charges <u>can</u> be described by $\rho(\vec{r})$



$\rho(\vec{r}) = q\,\delta^{(3)}(\vec{r}-\vec{r}\,')$, and $\vec{E}(r) = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$ as usual.

And then $\vec{\nabla}\cdot\vec{E} = \frac{q}{4\pi\varepsilon_0}\,\vec{\nabla}\cdot\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$

$= \frac{q}{\cancel{4\pi}\varepsilon_0}\cdot\cancel{4\pi}\,\delta^{(3)}(\vec{r}-\vec{r}\,')$

$= \rho(\vec{r})$ as it must be.

---

work it out! $\vec{\nabla}\cdot = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)$

- $\frac{\vec{r}-\vec{r}\,'}{|\ \ |^3} = \frac{(x-x')\,\hat{x} + (y-y')\,\hat{y} + \cdots}{[(x-x')^2 + (y-y')^2 + \cdots]^{3/2}}$

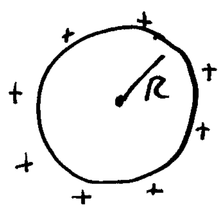
For $\vec{r}\neq\vec{r}\,'$, just <u>do</u> it!

you'll get zero ;)

Applying Gauss $\oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$ $\left(\text{or } \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \right)$

- Gauss' law is always true.
- Only useful to find E if symmetry lets you "pull it out"

Ex 1: Remember the horrid hw problem, uniform σ on sphere of radius R?



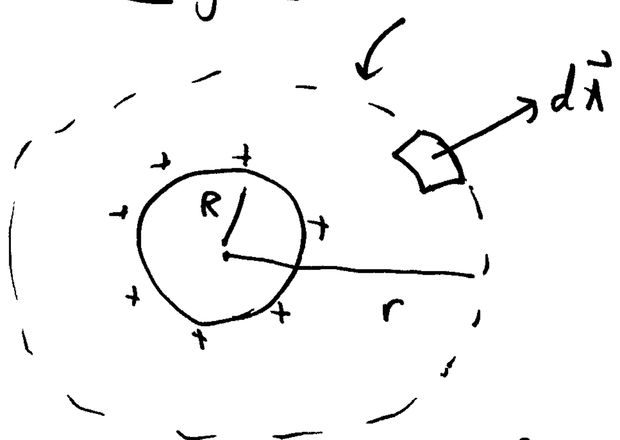
$\vec{E}(\vec{r}) = ?$

- Symmetry says $\vec{E}$ is radial. what other direction could it be?
- Symmetry says $|E(\vec{r})|$ depends only on r, not θ, or φ.

so $\vec{E} \cdot d\vec{A} = E\, dA$ ← Dot product $\Rightarrow \cos 0 = 1$.
Because $\vec{E}$ is radial, + so is $d\vec{A}$

wait!

WHAT is my $d\vec{A}$? what surface? It's not the sphere, it's imaginary ("Gaussian") sphere. I made it up!



So $\iint_{\text{imag sphere}} \vec{E} \cdot d\vec{A} = \iint E\, dA$

$= E \iint dA$

why? Because r = constant, so E = const, everywhere on that sphere.

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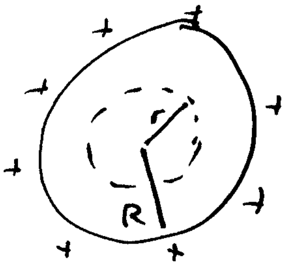
so $E \cdot 4\pi r^2 = \dfrac{Q_{enc}}{\varepsilon_0} \Rightarrow E = \dfrac{Q}{4\pi\varepsilon_0} \cdot \dfrac{1}{r^2}$ with $(Q = 4\pi R^2 \sigma)$

$\vec{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$, like a point charge @ origin.

what about inside? Same arguments!



$\vec{E}$ is radial, E depends only on R,

so $\oiint \vec{E} \cdot d\vec{A} = \iint E\, dA = E \iint dA = E \cdot 4\pi r^2$

But Q_{enc} is now zero. So $E = 0$!

---

what if that sphere had uniform ρ, rather than just σ?

Outside $\Rightarrow$ no difference, you can't tell! Only Q_{tot} matters

Inside $\Rightarrow$ $Q_{enc} \neq 0$ any more. What is it?

$$Q_{enc} = \underset{\substack{V \text{ inside} \\ \text{Gauss surf}}}{\iiint} \rho\, d\tau = \rho \cdot \iiint d\tau = \rho \cdot \frac{4}{3}\pi r^3$$

so $E \cdot 4\pi r^2 = \dfrac{1}{\varepsilon_0}\rho \cdot \dfrac{4}{3}\pi r^3 \Rightarrow \vec{E} = \left(\dfrac{\rho}{3\varepsilon_0} r\right)\hat{r}$

Vanishes only at origin!

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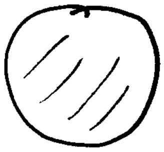
Basically 3 cases that are symmetric:

spherical geometry

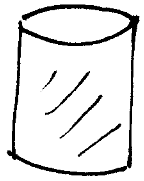


(Q's uniform in θ, φ, but maybe not in r!)

↑

Draw a spherical Gaussian surface.

cylindrical

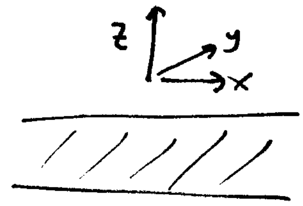


Q's uniform in θ, z But maybe not in r! (Better be infinite in z, then!)

↑

Draw a cylindrical "beer can" surface

planar



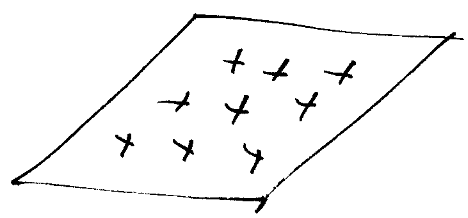
Q's uniform in x, y Maybe not in z Better be ∞ in x, y though.

↑

Draw a "pillbox" *

* Let's do one planar case.

We'll take a sheet with uniform σ (in x, y plane)



what's $\vec{E}(x, y, z)$?

This is clearly an idealization, but any flat surface will look like this nearby, so it'll be quite useful!

(1-2) -24

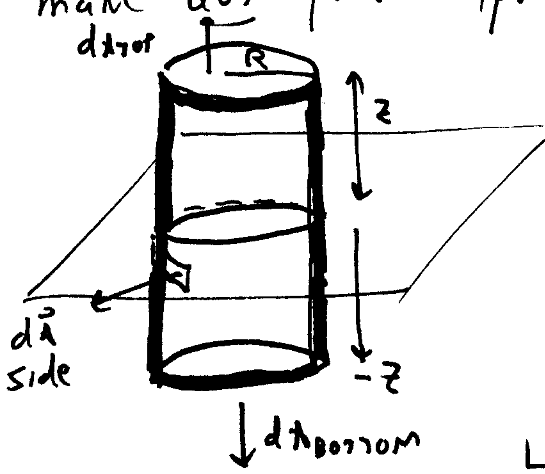
What imaginary surface do we pick?

- <u>Symmetry</u> says $\vec{E}(x,y,z)$ can only point in $\pm\hat{z}$ direction

Do you see why? Don't take my word for it!

- <u>Symmetry</u> says $\vec{E}(x,y,z) = E(z)\,\hat{k}$

This guides me: Draw a pillbox with faces $\perp$ or $//$ to $\hat{k}$ to make <u>dot</u> prod simple, <u>and</u> so E is uniform along surfaces.



There are other choices (e.g. 2.22 of Griffiths)

But look:

$$\underbrace{\oiint \vec{E}\cdot d\vec{A}}= Q_{enc}/\varepsilon_0$$

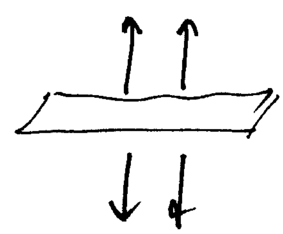
Do you see that?!

$$\underbrace{\iint_{TOP} \vec{E}\cdot d\vec{A}} + \iint_{\substack{\text{curvy}\\ \text{side}}} \vec{E}\cdot d\vec{A} + \iint_{Bottom} \vec{E}\cdot dA = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma\cdot\pi R^2}{\varepsilon_0}$$

$$E(z)\cdot\pi R^2 + \underbrace{0}_{\vec{E}\cdot d\vec{A}=0} + E(-z)\,\pi R^2 \underbrace{(-1)}_{\text{because } d\vec{A} \text{ points down}}$$

do you see why?

But $E(z) = -E(-z)$, by symmetry



so $\underset{\uparrow}{2\, E(z)\, \pi R^2} = \frac{\sigma}{\varepsilon_0} \pi R^2$

This is critical + subtle. L.H.S. is total flux = flux through top and through bottom. But both are equal, so sum is twice the flux thru top!

$$\vec{E}(z) = \frac{\sigma}{2\varepsilon_0} \hat{k} \quad \text{if } z>0$$

$$\vec{E}(z) = \frac{-\sigma}{2\varepsilon_0} \hat{k} \quad \text{if } z<0$$

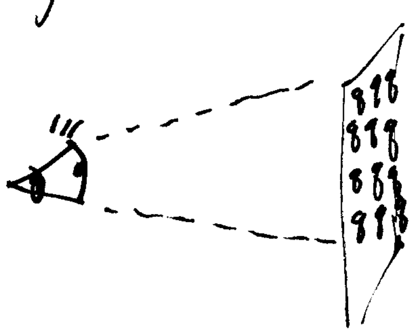
(you must think about that factor of 2!)

Notes: $\vec{E}$ is discontinuous at the sheet.

$\Delta E = \frac{\sigma}{\varepsilon_0} \hat{k}$ as move across. This turns out to be "deep" + universal!

→ E(z) is constant! Moving away from a sheet has no effect.

(How could you tell how far you are?)



As move away, $E_{\text{from each } q} \propto \frac{1}{r^2}$

But you "see" an area $\propto r^2$

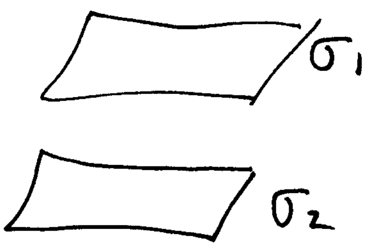
So effect of both cancels...

→ R of my imaginary surface Dropped out. Good!!

- There aren't many more such "easily" solved problems,

except: can use superposition

e.g

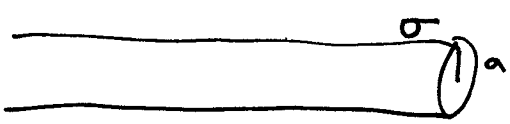


$\vec{E}$ everywhere is $\vec{E}$ due to σ_1
$+ \vec{E}$ due to σ_2.

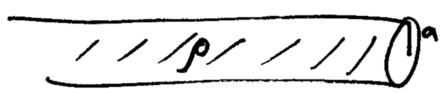
(etc)

For line:

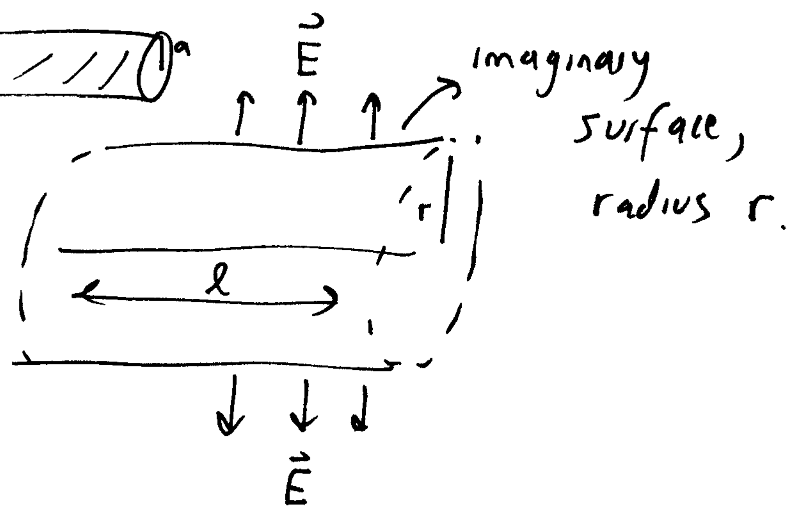
or rod: coated



or rod: filled



$\vec{E}$ outside is easy!



$$\iint E \cdot dA = \frac{Q_{enc}}{\varepsilon_0}$$

$\updownarrow$

$$\underbrace{E \iint dA}_{\text{Beer can}} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\lambda \cdot \ell}{\varepsilon_0} \left(\text{or } \frac{\sigma \cdot 2\pi a \ell}{\varepsilon_0}, \text{ or } \frac{\rho \cdot \pi a^2 \ell}{\varepsilon_0} \right)$$

$E \cdot 2\pi r \ell = \cdots$ (ℓ cancels, good! $E \sim \frac{1}{r}$, not $1/r^2$, not uniform)

3310 (1-2) - 27'

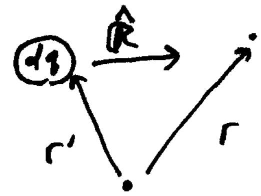
Maxwell's Eq'n #2 (Faraday's Law in Electrostatics)

$$\nabla \times \vec{E} = ?$$

We know $\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$ now.

what about $\vec{\nabla} \times E$? This adds <u>more</u> info about $\vec{E}$!

I know $d\vec{E}_{\text{from a pt charge } dq} = \dfrac{dq}{4\pi\varepsilon_0} \dfrac{\hat{R}}{R^2}$ with $\vec{R} = \vec{r} - \vec{r}\,'$



or, as we've been writing it:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\text{vol}} \rho(r')\, d\tau' \, \frac{\hat{R}}{R^2}$$

I could just <u>take</u> $\vec{\nabla} \times \vec{E}(\vec{r})$ and "grind":

It's a calc III problem, $\vec{\nabla} \times \dfrac{(\vec{r} - \vec{r}\,')}{(\vec{r} - \vec{r}\,')^2}$.

(Though we'll see there are other, more useful ways to see the result!)

(See next page!)

~~It's a Griffith's ch.1 problem 200 (1.13~~

And the result is... zero! (No matter what $\vec{r}\,'$ is, this function is not "curly")

$$\boxed{\vec{\nabla} \times \vec{E} = 0}$$

- $\vec{E}$ fields (in electrostatics!) are not curly. <u>certainly</u> true for a pt charge,

but $\vec{\nabla} \times (\vec{E}_1 + \vec{E}_2 + \ldots) = \vec{\nabla} \times \vec{E}_1 + \vec{\nabla} \vec{E}_2 + \cdots = 0 + 0 + 0 + \ldots = 0$ //

3310 (1-2) -28'

MATH INTERLUDE

Th #1: For any f, $\boxed{\vec{\nabla}\times(\vec{\nabla} f)=0}$ Reminds me of $\vec{A}\times\vec{A}=0$!

$\vec{\nabla} f$ points "up the hill", this $\vec{\nabla} f$ is a radial-like field, no curl!

[The proof: just do it! E.g $(\vec{\nabla}\times\vec{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial}{\partial y}A_x$
here, $(\vec{\nabla}\times(\vec{\nabla} f))_z = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0.$ etc.]

Th #2: $\vec{\nabla}\frac{1}{\mathscr{R}} = -\frac{1}{\mathscr{R}^2}\hat{\mathscr{R}}$ Reminds me of $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$.

[the proof: just do it! $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, write out $\vec{\mathscr{R}} = \vec{r} - \vec{r}\,'$
this is Griffiths HW prob. 1.13 b ...]

Now look: If $\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \rho(r')\,d\tau'\ \hat{\mathscr{R}}/\mathscr{R}^2$,

then Th #2 above says $\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \rho(r')\,d\tau'\ -\vec{\nabla}\left(\frac{1}{\mathscr{R}}\right)$

pull $\vec{\nabla}$ out of $\int d\tau'$ ($\vec{\nabla}$ has nothing to do with $d\tau'$!)

$\vec{E} = -\vec{\nabla}\left(\frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')\,d\tau'}{\mathscr{R}}\right) \equiv -\vec{\nabla} V(\vec{r})$ with $\boxed{V \equiv \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r')\,d\tau'}{\mathscr{R}}}$

and since $\vec{E} = -\vec{\nabla} V$

then $\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times(\vec{\nabla} V) = 0$ from Th. #1

$\boxed{\vec{\nabla}\times E = 0}$, always, in electrostatics

$\vec{E}$ is "radial-like", has no curl to it.

3310 (1-2) - 29.

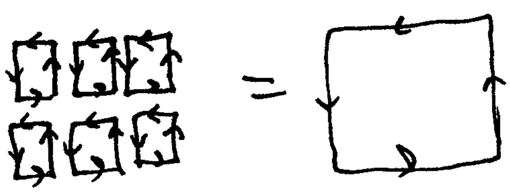
· Another Math Interlude: Stoke's Theorem (Griff 1.3.5)

$$\iint_{\text{Any open } S} (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \oint_{\text{Line around } S} \vec{F} \cdot d\vec{L} \quad (\text{for } \underline{\text{any}}\ \vec{F})$$

again, like $\int \frac{df}{dx}\, dx = f(b) - f(a)$, ("integral of deriv = fn at boundary")

In words, $\vec{\nabla} \times \vec{F}$ is circulation or swirl at a point.

If add up all the swirls ⇒ "total swirl", this is just what the circulation $\oint v \cdot dL$ around outside gives



If fish swims around edge of pond and has a current with it the whole way, there must be "whirlpools" somewhere in the middle! (Maybe it's in a toilet?)

So if $\vec{\nabla} \times \vec{E} = 0$, then $\oint_{\text{Any loop}} \vec{E} \cdot d\vec{L} = 0$ by Stoke's th!

⇒ Maxwell's Eq'n in electrostatics (Faraday's Law)

$$\oint \vec{E} \cdot d\vec{L} = 0 \quad \text{or} \quad \vec{\nabla} \times \vec{E} = 0$$

[Both say same thing, by Stoke's.]

↳ Griff worked from here, showing $\int \left(\frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}\right) \cdot d\vec{l} = 0$ for any path.

3310 (1-2) – 30.

One more math interlude: fund. theorem of calc:

$$\boxed{\int_A^B (\vec{\nabla} F)\cdot d\vec{L} = F(B) - F(A)}$$

(like: $\int_A^B \frac{df}{dx}\,dx = f(b) - f(a)$)

But, Since $\vec{E} = -\vec{\nabla} V$ we have

$$-\int_A^B \vec{E}\cdot d\vec{L} = +\int_A^B \vec{\nabla} V \cdot d\vec{L} = +\left(V(B) - V(A)\right)$$

- V is a scalar fn, the "potential". (Not Pot. Energy, we'll get to that!)
  It's a number at every point in space.

- For a point charge, $\boxed{V = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{\mathscr{R}}}$ (Because $\rho \to q\,\delta^{(3)}$)

  Except, (ambiguity!) you could always add constant (to it),
  Because $\vec{E} = -\vec{\nabla} V$ would not be changed.

- Usually choose $V(\vec{r}\to\infty) = 0$ to set the value of this constant

  so $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$

  q at origin

  By superposition, $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')\,d\tau'}{|r-r'|}$, (as we had before)
  with $\rho(r')$

- $\oint \vec{E}\cdot d\vec{L} = \iint (\nabla\times\vec{E})\cdot d\vec{A} = 0.$ ($\vec{E}$ is "conservative".)

Summary so far: If define $V(A) = 0$

$$V(r) = -\int_A^r \vec{E}\cdot d\vec{L}$$ ← Find V from $\vec{E}$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(r')\,d\tau'}{|r-r'|}$$ ← or, Find V from ρ.

$$\vec{E} = -\vec{\nabla}V$$ ← Find $\vec{E}$ from V

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(r')\,d\tau'}{(r-r')^3}(\vec{r}-\vec{r}')$$ ← or, Find $\vec{E}$ from ρ

• The integral for V is easier, so we may want (prefer!) to find V, + then $\vec{E} = -\vec{\nabla}V$ is a "1-stepper".

Finally, $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(-\vec{\nabla}V) = -\nabla^2 V$

But $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$

so $$\boxed{\nabla^2 V = -\rho/\varepsilon_0}$$ ← Poisson's eq'n.
It's really Gauss + $\vec{\nabla}\times\vec{E} = 0$ combined!!

This is all we really need. Given ρ, solve for V, then get $\vec{E}$, yay!

If empty space

$$\boxed{\nabla^2 V = 0}$$ ← Laplace's Eq'n.
We'll spend a lot of time learning tricks to solve this, often with no integrals needed.

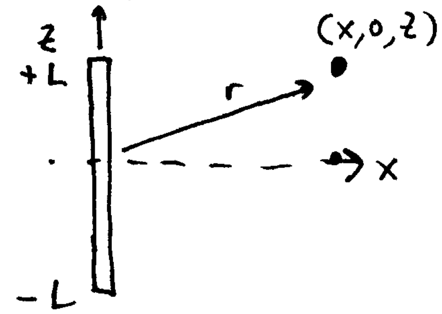
3310 (1-2) -32.

Examples : let's find $V(r)$

Case 1: Point charges: $V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$

superposition, if several.

Case 2:



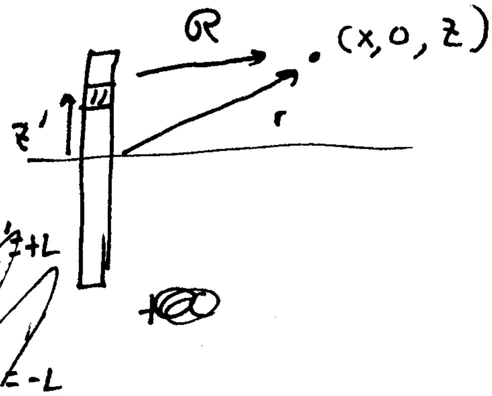
Uniform Line of charge, density λ, finite length, $-L$ to $+L$.

Find $\vec{E}(\vec{r})$?

Hard! Gauss' law: no good, no symmetry. What surface would you draw? ($\vec{E}$ not constant on any obvious surface)

Could integrate, but let's try using/find $V(\vec{r})$ instead.

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{-L}^{+L}\frac{\lambda\, dz'}{\sqrt{x^2+(z-z')^2}}$$



$$\cancel{= \frac{\lambda}{4\pi\varepsilon_0}\log\left(z'-z+\sqrt{x^2+(z-z')^2}\right)\Big|_{z'=-L}^{z'=+L}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\log\frac{L+z+\sqrt{x^2+(L+z)^2}}{+L-z+\sqrt{x^2+(L-z)^2}}$$

(could always add $+C$, but no need, since $V(x\to\infty)\to 0$

(Take $\frac{\partial}{\partial x}$ to Find E_x, etc)

<u>Last Example</u> Sometimes you <u>know</u> $\vec{E}$ already.

Then V is even easier! $V(r) - V(a) = -\int_a^r \vec{E}\cdot d\vec{L}$.

Consider our "<u>Shell</u> of σ" : $E = \frac{1}{4\pi\epsilon_0}\, q/r^2 \,\hat{r}$ outside

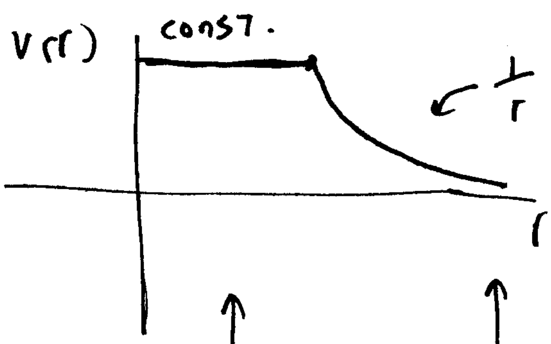
$E = 0$ inside.

<u>Define</u> $V(\infty) = 0$

$$V(r>R) \underset{-0}{=} -\int_\infty^r \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' = -\frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r'}\right)_\infty^r = +\frac{q}{4\pi\epsilon_0 r}$$

$$V(r<R) - V(R) = -\int_R^r 0\cdot dr' = 0.$$

so $V(r<R) = V(R) = q/4\pi\epsilon_0 R$



$-\nabla V = 0$ as it should be

$-\nabla V = -\nabla \frac{1}{r} = +\frac{1}{r^2}$ as it should be.