Why potentials can be easier to work with than fields:

- V is a scalar!
- V is never discontinuous
- Often easy to calculate from charge density.

in gory detail: Griffiths p.88-90

Example: spherical shell of charge Q, radius R:

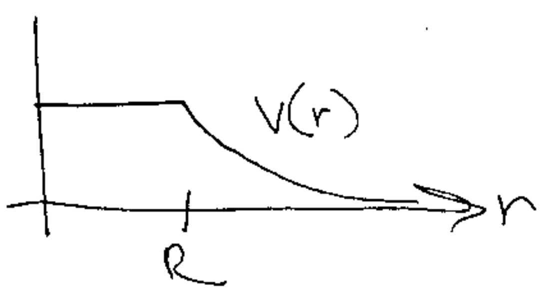
$$\vec{E} = \frac{q\hat{r}}{4\pi\varepsilon_0 r^2} \quad r>R$$

$$= 0 \quad r<R$$

$$\text{So } V = -\int_{\infty}^{r} d\vec{l}'\cdot\vec{E}(\vec{r})$$

$$= \frac{-1}{4\pi\varepsilon_0}\int_{\infty}^{r} dr'\,\frac{Q}{r'^2} \quad \text{for } r>R \Rightarrow V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

$$= \frac{-1}{4\pi\varepsilon_0}\int_{\infty}^{R} dr'\,\frac{Q}{r'^2} \quad \text{for } r<R \Rightarrow V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$$



Griffiths' "eternal triangle": differential & integral relationships between $\rho, \vec{E}, V$:

A slightly different perspective: shows hierarchy more clearly.

$$\rho$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{r^2}\rho\, d^3r \quad \downarrow\ \uparrow \quad \rho = \varepsilon_0\, \mathrm{div}\vec{E}$$

$$\vec{E}$$

$$V = -\int \vec{E}\cdot d\vec{l} \quad \downarrow\ \uparrow \quad \vec{E} = -\vec{\nabla} V$$

$$V$$

$$V = \frac{1}{4\pi\varepsilon_0}\int d^3r \frac{\rho}{r} \qquad \rho = -\varepsilon_0 \nabla^2 V$$

The energy in the electric field:

Clearly, moving a charge in an electric field will involve doing work (unless normal to $\vec{E}$):

$$\vec{F} = Q\vec{E} = Q(-\vec{\nabla}V) = -\vec{\nabla}(QV) = -\vec{\nabla}U$$ ← Potential energy

So for a point charge, $U = QV = W$: work required to bring charge to this position from $V=0$.

What about the work needed to create V?

Start with 1 point charge: $W_1 = 0$ (ignoring self energy)

$W_2 = Q_2 \cdot V_1 = Q_2 \frac{Q_1}{4\pi\varepsilon_0 r_{12}}$ (same as if I reverse labels 1, 2.)

Add a third:

$$W_3 = Q_3 V_{1+2} = Q_3 \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1}{r_{13}} + \frac{Q_2}{r_{23}}\right]$$

$W_{tot} = W_1 + W_2 + W_3$ = energy of this charge configuration

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1 Q_2}{r_{12}} + \frac{Q_1 Q_3}{r_{13}} + \frac{Q_2 Q_3}{r_{23}}\right]$$

For N charges: $$= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{Q_i Q_j}{r_{ij}}$$

↳ Avoids double-counting $\frac{Q_1 Q_2}{r_{12}}$ and $\frac{Q_2 Q_1}{r_{21}}$

...or, can deliberately double-count and divide by 2:

$$W = \frac{1}{8\pi\varepsilon_0} \sum_{i=1}^{N} Q_i \underbrace{\sum_{j=1 \neq i}^{N} \frac{Q_j}{r_{ij}}}_{= V \text{ that } Q_i \text{ "sees"}}$$

$$W = \frac{1}{2} \sum_{i=1}^{N} Q_i \underbrace{V(\vec{r}_i)}_{\text{from all other charges.}}$$

W is also the energy stored in the charge configuration.
Can see ~~that~~ the energy is actually stored in the field:
Take continuous-charge analogue:

$$W = \frac{1}{2}\int d^3r \, \rho(\vec{r}) V(\vec{r})$$

⇒ Note - only need to integrate in regions where $\rho \neq 0$.