P3310 LECTURE 6

Symmetry and Gauss's Law: $\oint_S \vec{E}\cdot d\vec{A} = \frac{1}{\varepsilon_0} Q_{enclosed}$

Take a spherically symmetric charge distribution $\rho(\vec{r}) = \rho(r)$.

$\vec{E}(\vec{r}) = ?$

By symmetry, can know field depends only on r, not ϕ or θ.
Also, know field can have no ϕ, θ component.

Therefore take sphere of radius r_0 as Gaussian surface:

$$\oint_S \vec{E}\cdot d\vec{A} = E(r_0)\cdot \text{Area}$$ since $\vec{E}$ is always normal to the surface

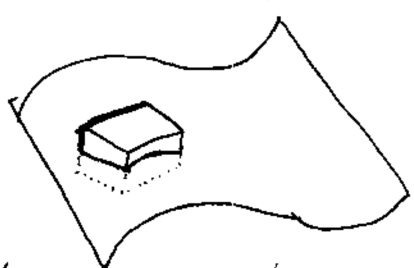
$$\frac{1}{\varepsilon_0} Q_{enc} = \int_0^{r_0} dr\, 4\pi r^2 \rho(r) = E(r_0)\cdot 4\pi r_0^2$$

$$\Rightarrow \vec{E}(r_0) = \frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{r_0^2}\hat{r}$$

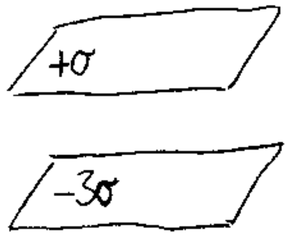
So $\vec{E}(r_0)$ is same as if all charge <u>inside r_0</u> were concentrated at the origin, and all (spherically symmetric) charge <u>outside</u> r_0 were zeroed.

Can exploit other symmetries (cylindrical, plane, etc) as well.

Griffiths Fig. 2.20:

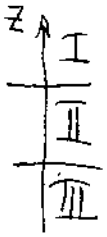


Can we use the "pillbox" here to determine $\vec{E}$? No! No symmetry, so can only use Gauss's law to get the flux through the box.
If the "flying carpet" were a flat plane, it would be symmetric.



Two infinite planes of charge:

What is field in each region?



Region I: $\uparrow\frac{\sigma}{2\varepsilon_0} + \downarrow\frac{3\sigma}{2\varepsilon_0} = -\frac{\sigma}{\varepsilon_0}$

Region II: $\downarrow\frac{\sigma}{2\varepsilon_0} + \downarrow\frac{3\sigma}{2\varepsilon_0} = -\frac{2\sigma}{\varepsilon_0}$

Region III: $\downarrow\frac{\sigma}{2\varepsilon_0} + \uparrow\frac{3\sigma}{2\varepsilon_0} = +\frac{\sigma}{\varepsilon_0}$

Potentials

The static electric field has zero curl everywhere.
(called an "irrotational" field.)

If a vector field $\vec{V}$ has no curl, then:

$\oint d\vec{l}\cdot\vec{V} = 0$ on any closed path. (obvious from Stokes's thm)

$\Rightarrow \int_{\vec{a}}^{\vec{b}} d\vec{l}\cdot\vec{V}$ depends only on endpoints, not path

Recall that these are two properties of the gradient of a scalar field $f(\vec{r})$:

$$\int_{\vec{a}}^{\vec{b}} d\vec{l}\cdot(\vec{\nabla} f) = f(b) - f(a)$$

If $\vec{V}$ has this property under integration, it is $\vec{\nabla} f$.

Note that $\vec{V} = \vec{\nabla} f$ completely determines $\vec{V}(\vec{r})$ but $\vec{\nabla} f = \vec{V}$ does <u>not</u> uniquely determine $f(\vec{r})$ since $f(r) \to f(r) + K$ does not affect gradient. Call $f(\vec{r})$ the <u>scalar potential</u> of $\vec{V}(\vec{r})$

Static electric field is the gradient of scalar electric potential (NOT potential energy): $\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$

This means: $\int_{\vec{a}}^{\vec{b}} \vec{E}(\vec{r}) \cdot d\vec{l} = V(\vec{a}) - V(\vec{b})$ note sign convention!

So this scalar function $V(\vec{r})$ completely determines vector field $\vec{E}(\vec{r})$. (Perhaps unsurprising, since $\vec{E}$ is the result of a configuration of scalar charges.) Note also that $\nabla^2 V(r) = \frac{-\rho}{\epsilon_0}$ (Poisson eqn)

Ambiguity: Since you can add a constant to $V(\vec{r})$ and not change its gradient (sort of like an origin to a coordinate system), need to decide where to set $V(\vec{r})=0$. Usually at $\vec{r}=\infty$. (except when charge distribution goes to $r=\infty$.)

Superposition: can add potentials from multiple charges to find the total potential (as with $\vec{E}$ — not surprising since $\vec{\nabla}(a(\vec{r})+b(\vec{r})) = \vec{\nabla}a + \vec{\nabla}b$).

How to calculate $V(\vec{r})$: since $V(\vec{a}) - V(\vec{b}) = \int_{\vec{a}} d\vec{l} \cdot \vec{E}(\vec{r})$, take $V(\infty)=0$: $V(\infty) - V(\vec{b}) = \int_{\infty}^{\vec{b}} d\vec{l} \cdot \vec{E}(\vec{r})$.

For a point charge at the origin, then,

$$V(\vec{r}) = \frac{-1}{4\pi\epsilon_0} \int_{\infty}^{r} dr' \frac{q}{r'^2} = \frac{-1}{4\pi\epsilon_0} \frac{q}{r'}\Big|_{\infty}^{r} = \boxed{\frac{1}{4\pi\epsilon_0} \frac{q}{r}}$$

So + charge has + potential. Superposition says:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq(\vec{r}')}{|\vec{r}-\vec{r}'|}$$