P3310 LECTURE 5

- CLASS survey: see web site
- Solutions will be up this PM

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}$$: Coulomb's Law in SI dimensions.

What is ε_0? Really a unit conversion constant (sort of like Boltzmann's constant, that relates energy to temperature units)

Also: recall $\vec{\mathscr{r}} \equiv \vec{r}-\vec{r}'$.

Force is actually a function of the <u>electric field</u>:

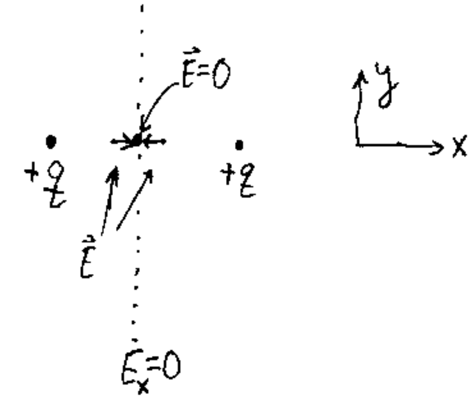
$\vec{F} = Q\vec{E}$ where $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}$ and q is at $\vec{r}'$!

Superposition: fields from multiple charges add as vectors:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{\text{charges}}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i \quad \text{where } q_i \text{ is at } \vec{r}_i'$$

$\vec{E}$ is <u>not</u> a mathematical convenience: it is as "real" as charge or mass. $\vec{E}$ has momentum and energy independent (sort of) of charges.

Multiple-charge problems can often be made easier by exploiting symmetries:



Most of the time we will be dealing with continuous charge distributions, not finite point charges:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\mathscr{r}}\, dq \longleftarrow$$ Griffiths-ese for an integral over charge.

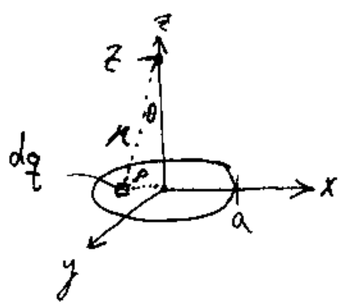
$\lambda\, dl$ where $\lambda = \frac{\text{charge}}{\text{length}}$

$\sigma\, dA$ where $\sigma = \frac{\text{charge}}{\text{surface area}}$

$\rho\, d^3r$ where $\rho = \frac{\text{charge}}{\text{volume}}$

Let's call charge density ρ_q and cylindrical radius ρ_c when there is a possibility of confusion.

Example: A circular plate of uniform charge density σ, centered on x-y plane, with radius a: what is $\vec{E}$ on z axis?



By symmetry, $\vec{E} \parallel \hat{z}$ so only calculate this component.

$$d\vec{E} = \left(\frac{1}{4\pi\varepsilon_0}\cos\theta \frac{1}{\mathscr{r}^2}\sigma\rho\, d\phi\, d\rho\right)\hat{z}$$

$$\underline{\mathscr{r}^2 = \rho^2 + z^2, \quad \cos\theta = \frac{z}{\sqrt{\rho^2+z^2}}}$$

$$\vec{E} = \hat{z}\int_0^a d\rho \int_0^{2\pi} d\phi \left[\frac{\sigma}{4\pi\varepsilon_0}\frac{z}{\sqrt{\rho^2+z^2}}\frac{1}{\rho^2+z^2}\rho\right] = \frac{2\pi z\sigma}{4\pi\varepsilon_0}\hat{z}\underbrace{\int_0^a d\rho \frac{\rho}{(\rho^2+z^2)^{\frac{3}{2}}}}_{= \frac{1}{\sqrt{\rho^2+z^2}}}$$

$$= \frac{z\sigma}{2\varepsilon_0}\left(\frac{-1}{\sqrt{a^2+z^2}} + \frac{1}{|z|}\right)$$

Does this behave right in limiting cases?

Take $\left|\frac{z}{a}\right| \ll 1$: $E_z \to 0 + \frac{\sigma}{2\varepsilon_0}$ $z>0$

$\to -\frac{\sigma}{2\varepsilon_0}$ $z<0$

Take $\left|\frac{z}{a}\right| \to \infty$: use $\lim_{z\to\infty}\left(\frac{-1}{\sqrt{a^2+z^2}} + \frac{1}{|z|}\right) = \frac{a^2}{2z^3}$

so $|E_z| \to \frac{\sigma a^2}{4z^2\varepsilon_0}$ which looks like field of a point charge:

$$E_q = \frac{q}{4\pi\varepsilon_0 z^2} = \frac{\sigma a^2}{4z^2\varepsilon_0} \Rightarrow q = \sigma\pi a^2$$ which is the disc charge.

(important corollary: Electric field near a sheet of constant charge density σ is a constant $E = \frac{\sigma}{2\varepsilon_0}$.

Divergence, curl of $\vec{E}$: For static charges $\operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0}$, $\operatorname{curl}\vec{E} = 0$.

(Griffiths treats this as ~~a~~ result; it can be treated as the first principle of electrostatics.)

$\operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0}$ is Gauss's Law. ($\operatorname{div}\vec{E} = 4\pi\rho$ in Gaussian units.)

Divergence theorem relates this to the integral form:

$$\oint_S d\vec{A}\cdot\vec{E} = \int_V d^3r\,\mathrm{div}\,\vec{E} = \frac{1}{\varepsilon_0}\int_V d^3r\,\rho(\vec{r}) = \frac{1}{\varepsilon_0}Q_{enclosed}.$$

<u>Very</u> powerful when there are symmetries to **help you!**

in general, call $\int_S \vec{V}\cdot d\vec{A}$ the "flux" of vector field $\vec{V}$ through surface S. Gauss's Law tells us flux of $\vec{E}$ through a <u>closed</u> surface is equal to $\varepsilon_0 Q_{enc}$.

Consistency check: Recall $E(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int d^3r'\,\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}$

$\mathrm{div}\,\vec{E}$ is therefore $\mathrm{div}\left[\dfrac{1}{4\pi\varepsilon_0}\displaystyle\int d^3r'\,\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}\right]$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\underbrace{\mathrm{div}\left(\frac{\rho(r)\,\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}_{=4\pi\,\delta^3(\vec{\mathscr{r}})\,\rho(\vec{r}')}$$

$$\mathrm{div}\,\vec{E} = \frac{\rho(\vec{r})\cancel{4\pi}}{\cancel{4\pi}\varepsilon_0}\int d^3r'\,\delta^3(\vec{\mathscr{r}}') = \frac{\rho(\vec{r})}{\varepsilon_0}$$