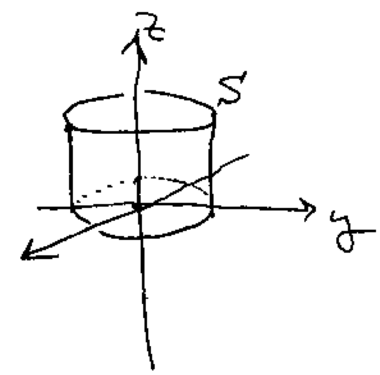
New (bad) math: Dirac delta function

Recall divergence calculation last lecture: $\text{div}\,\vec{V}$ where $\vec{V} = \frac{1}{\sqrt{x^2+y^2}}\,\hat{\rho}$

We found $\text{div}\,\vec{V} = \frac{2}{x^2+y^2} - \frac{2}{(x^2+y^2)^2}(x^2+y^2) = 0$ if $x^2+y^2 \neq 0$.

Yet if we use divergence theorem and calculate $\int_S \vec{V}\cdot d\vec{A}$

with S a cylinder from $z=0$ to $z=1$, radius 1, centered on z axis:



$d\vec{A}\bullet\vec{V} = 1$ on cylinder wall

$= 0$ on end caps.

So $\int_S \vec{V}\cdot d\vec{A} = \int_0^1 dz \int_0^{2\pi} d\phi\; 1 = 2\pi$

But according to divergence theorem, $2\pi = \int_0^1 dz \int_0^1 d\rho \int_0^{2\pi} d\phi\; \text{div}\,\vec{V}(\vec{r})$

If $\text{div}\,\vec{V} = 0$ everywhere except z-axis but its integral is finite, then $\text{div}\,\vec{V} = \infty$ at z-axis.

First encounter with an object called the Dirac delta function:

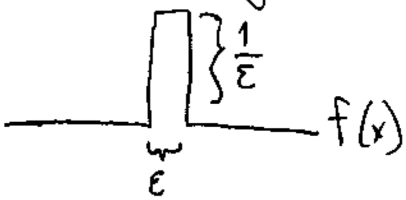
The deltafunction $\delta(x) = 0$ if $x \neq 0$

$= \infty$ if $x = 0$

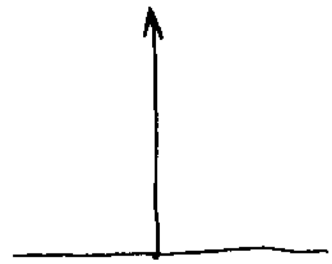
$$\int_{0-\epsilon}^{0+\epsilon} \delta(x)\,dx = 1$$

Not finite everywhere → not legally a "function" (but that won't stop us)

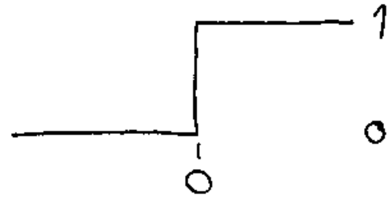
Several ways to picture $\delta(x)$:



$\lim_{\varepsilon \to 0}$ $\longrightarrow$

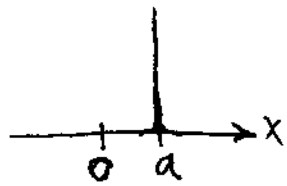


or derivative of a step function:



You can put the "spike" somewhere besides zero.

$\delta(x-a)$:



Very important property of $\delta(x)$: For any function $f(x)$:

$$\int dx\, f(x)\delta(x) = f(0) \Rightarrow \int dx\, f(x)\delta(x-a) = f(a)$$

$$\left(\text{indeed } f(x)\delta(x) = f(0)\,\delta(x)\right)$$

So, for example, $\displaystyle\int_{-\infty}^{\infty} dx\, \frac{x^3}{7}\,\delta(x-2) = \frac{8}{7}$.

In 3 dimensions: $\delta^3(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$

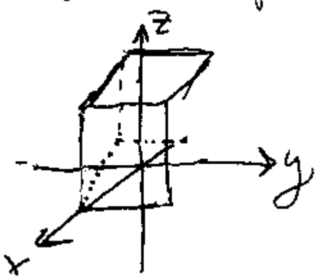
$$\text{so } \int_{\text{all space}} d^3r\, \delta^3(r) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\ \delta(x)\delta(y)\delta(z)$$

$$= \int dx\, \delta(x) \int dy\, \delta(y) \int dz\, \delta(z) = 1 \cdot 1 \cdot 1 = 1$$

(... and integral doesn't have to be over all space, just include the origin.)

Divergence theorem paradoxes can now be resolved quantitatively: Griffiths p. 50 does it for the 3-D $\text{div}\left(\frac{1}{r^2}\hat{r}\right)$ problem. Let's do it for our cylindrical problem $\vec{V} = \frac{1}{\rho}\hat{\rho}$:

We found that $\int_S \vec{V}\cdot d\vec{A} = 2\pi$ for integration over surface from $z=0$ to 1, ρ out to 1. This means that $\int d^3r \, \text{div}\vec{V} = 2\pi$ over this volume. Let's make the integration easier by integrating over a cube (the div is zero away from the z axis so this doesn't change the integral anyway):



$z: (0, 1)$
$y: (-\frac{1}{2}, \frac{1}{2})$ ← these bounds don't matter
$x: (-\frac{1}{2}, \frac{1}{2})$ ← as long as they include 0.

know that $\displaystyle\int_0^1 dz \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \, \text{div}\vec{V} = 2\pi$

$$\longrightarrow \text{div}\,\vec{V} = 2\pi\,\delta(x)\delta(y) = 2\pi\,\delta^2(\rho).$$

3-D equivalent: $\text{div}\left(\frac{1}{r^2}\hat{r}\right) = 4\pi\,\delta^3(\vec{r})$

... and more generally, if the source is at $\vec{r}\,'$ position:

$$\text{div}\left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}\,\frac{1}{|\vec{r}-\vec{r}\,'|^2}\right) = 4\pi\,\delta^3(\vec{r}-\vec{r}\,')$$

Griffiths compact notation uses script $\mathscr{r}$ for this separation vector, so:

$$\text{div}\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\,\delta^3(\vec{\mathscr{r}})$$ careful handwriting: $\mathscr{r} \neq \mu$

Electrostatics: physics of electric charges at rest.

Start from Coulomb's Law: force between two point charges: Q and q, at positions $\vec{r}$ and $\vec{r}\,'$:

$$\vec{F} = k\,\frac{qQ}{\mathscr{r}^2}\,\hat{\mathscr{r}} \qquad \left(\text{recall } \vec{\mathscr{r}} = \vec{r} - \vec{r}\,'\right)$$

k is a constant that depends on the dimensional conventions: there are several "unit systems" for E&M. But unlike mechanics, these systems don't agree on the basic dimensions (not just units) of the electric fields & charges!

Gaussian system: uses cgs mechanical units (length cm, mass g) and derives unit of charge so that $k=1$:

$$\vec{F} = \frac{qQ}{\mathscr{r}^2}\,\hat{\mathscr{r}} \rightarrow \text{so charge has dimensions of } \left(\sqrt{\text{force}}\right)(\text{length}) \rightarrow \text{called esu or statcoulomb}$$

SI: uses mks mechanical units (length m, mass kg) and an independently obtained unit of charge called coulomb (C). This requires a derived constant to relate mechanical & electric units: $k = \frac{1}{4\pi\varepsilon_0}$ where $\varepsilon_0 = 8.85\times10^{-12}\,\frac{C^2}{N\cdot m^2}$

SI introduces (in my opinion, needless) complications to keep familiar electrical engineering units like volts, amps, coulombs in the equations. But Griffiths uses them so for the most part we will too.