P3310 LECTURE 37

From last lecture: force and torque on a magnetic dipole with moment $\vec{m}$:

$$\text{Force} = -\text{grad}(\vec{m}\cdot\vec{B})$$

$$\text{Torque} = \vec{m}\times\vec{B}$$

Note that $\vec{F} = -\text{grad}(\vec{m}\cdot\vec{B})$ implies that the energy of the system is $\vec{m}\cdot\vec{B}$:

$\vec{m}\cdot\vec{B}$ is a scalar field, and $-\text{grad}\, U = \vec{F}$.

Given the information from electrically polarized material, we can define $\vec{M}$ = <u>magnetization</u> ≡ magnetic dipole moment per unit volume.

Materials acquire a magnetization in response to magnetic fields — but in very different ways depending on the material. Magnetization is aggregate effect of microscopic currents in material; loops are at molecular scale:

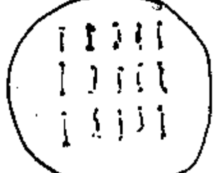
Paramagnetic and diamagnetic materials: linear response of atomic electron spins & orbits to an external field

Ferromagnetic materials: highly nonlinear, capable of large permanent magnetization. Details later.

Magnetization & Polarization parallels:

Const. $\vec{P}$:

$\vec{P}\uparrow$

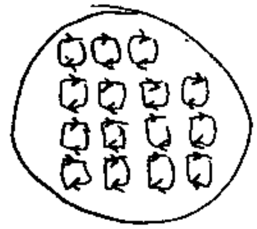


Each $\pm$ is a dipole

$\Rightarrow$ all cancel since + and − charges are together — except at edge, where net pileup of unpaired ends $\Rightarrow$ bound charge

constant $\vec{M}$:

$M \odot$



Adjacent microscopic dipoles have mutually canceling fields:

I up & down

... except at boundary, where we find bound current −

Field of magnetized object: start with vector potential

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

where $\vec{m}$ is at origin

So if dipole is at $\vec{r}'$ instead, need to subtract $\vec{r}'$ from $\vec{r}$:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

For an extended object, need to integrate over it:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{M}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

Note: Griffiths 2nd edition uses very muddled notation here 3rd ed. is <u>much</u> better.

Now, use identity $\text{grad}\frac{1}{r} = \frac{\hat{r}}{r^2}$, applied to $\hat{\mathscr{r}}$:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \left[\vec{M}(\vec{r}')\times \text{grad}'\frac{1}{\mathscr{r}}\right]$$

Now, integrate by parts, using identity

$$\text{curl}\left(f(\vec{r})\vec{V}(\vec{r})\right) = f\ \text{curl}\,\vec{V} - \vec{V}\times \text{grad}\, f\ :$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left\{\int d^3r'\ \frac{1}{\eta}\,\text{curl}'\,\vec{M}(\vec{r}\,') - \int d^3r'\ \text{curl}'\,\frac{\vec{M}(\vec{r}\,')}{\eta}\right\}$$

$$= \underbrace{\frac{\mu_0}{4\pi}\int d^3r'\ \frac{1}{\eta}\,\text{curl}'\,\vec{M}(\vec{r}\,')}_{\substack{\text{Potential of a}\\ \text{volume current}\\ \vec{J}_b = \text{curl}\,\vec{M}}} + \underbrace{\frac{\mu_0}{4\pi}\oint \frac{1}{\eta}\left[\vec{M}(\vec{r}\,')\times d\vec{A}'\right]}_{\substack{\text{Surface containing the magnetized}\\ \text{material : Looks like surface}\\ \text{current potential, with } \vec{K}_b = \vec{M}\times\hat{n}}}$$

So $$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\ \frac{\vec{J}_b(\vec{r}\,')}{\eta} + \frac{\mu_0}{4\pi}\oint_S \frac{\vec{K}_b(\vec{r}\,')}{\eta}\,dA'$$

Compare to electrostatic polarization: $\rho_b = -\text{div}\,\vec{P}$, $\sigma_b = \vec{P}\cdot\hat{n}$

Example: uniformly magnetized sphere (very cute): radius R, magnetization constant $\vec{M}$:

$\vec{J}_b = \text{curl}\,\vec{M} = 0 \implies$ no volume current in bulk

$\vec{K}_b = \vec{M}\times\hat{n} = \vec{M}\times\hat{n} =$ Now, $|\vec{M}\times\hat{n}| = M\sin\tilde{\theta}$

where $\tilde{\theta}$ = angle btw $\vec{M}$ and $\hat{r}$

$\rightarrow$ that's just coordinate θ.

$\hat{z}\times\hat{r}$ is in ϕ direction

so $\vec{K}_b = M\sin\theta\,\hat{\phi}$

This is the same current as a shell of const. surface charge density! $\vec{K} = \sigma \vec{v} = \sigma \omega R \sin\theta \, \hat{\phi}$

$\Longrightarrow$ same behavior: $\vec{B}$ constant $= \frac{2}{3}\mu_0 \vec{M}$ inside, pure dipole field outside, with dipole moment $\vec{M} \cdot \text{Volume} = \frac{4}{3}\pi R^3 \vec{M}$.