NOTE. PROBLEM SET 10 DUE FRIDAY

Dipoles & quadrupoles: electric vs. magnetic

— Multipole expansion at large radius: for a single current loop:

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} r^{-(n+1)} \oint d\vec{l}' \; r'^n P_n(\cos\tilde{\theta})$$

Generalizing to arbitrary current distribution:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} r^{-(n+1)} \int d^3r' \, r'^n \vec{J}(\vec{r}') P_n(\cos\tilde{\theta})$$

Dipole potential: $\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi r^2}(\vec{m}\times\hat{r})$

Notation: Magnetic dipole moment is often denoted by $\vec{\mu}$, especially in quantum mechanics.

Consider $\vec{m} = m_0\hat{z}$:

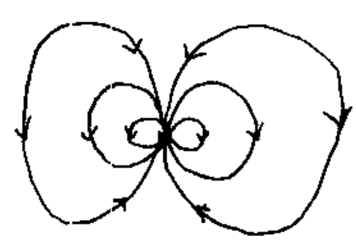
$$\vec{A}(\vec{r}) = \frac{\mu_0 m_0}{4\pi r^2}(\hat{z}\times\hat{r}) = \frac{\mu_0 m_0}{4\pi r^2}\sin\theta\,\hat{\phi}$$

Now, find $\vec{B} = \text{curl}\,\vec{A}$:

$$= \frac{\mu_0 m_0}{4\pi}\left\{\frac{\hat{r}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^2}\right)\right] - \frac{\hat{\theta}}{r}\frac{\partial}{\partial r}\frac{\sin\theta}{r}\right\}$$

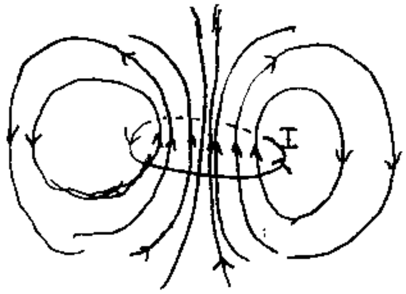
$$= \frac{\mu_0 m_0}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

... which is a pure dipole field (as in spinning charge sphere in Lecture 34 notes). Field lines look like ideal dipole:

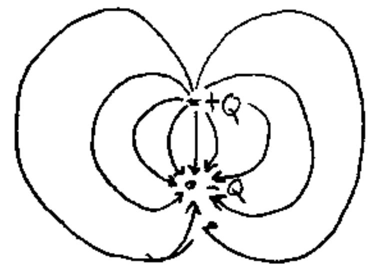


Up close, a physical magnetic dipole is very different:

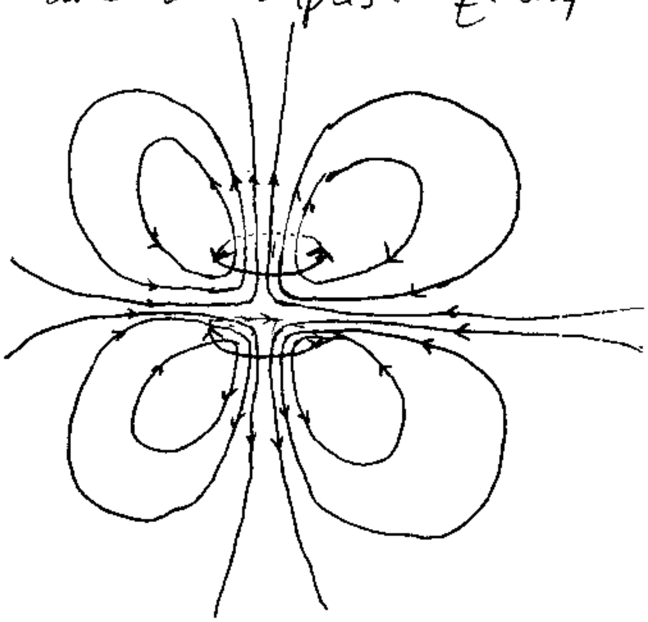
Magnetic

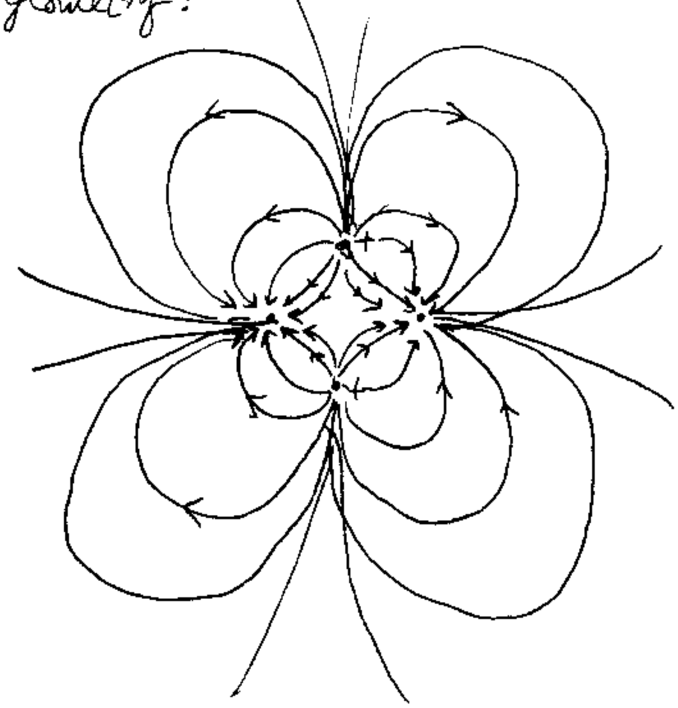


Electric



Quadrupoles: in Magnetostatics, two loops with opposite current are the simplest quadrupole geometry:

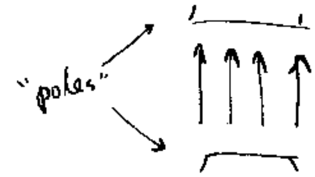




Again, same field at large distances, different up close.

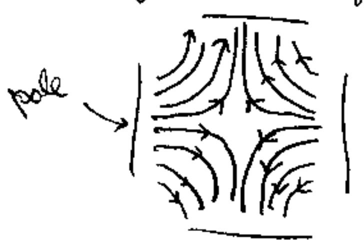
Should not be surprising! Both fields, in regions of no charge or current, are Laplacian fields with zero div and curl. Just as we can use a magnetic scalar potential in these regions, we can also — if we're nuts — define an electric vector potential so $\vec{E} = \text{curl}\,\vec{V}$!) So it's unsurprising that the fields can be expanded in the same series.

A side note on terminology: people often refer to a "dipole magnet" and mean the field up close: (for electromagnets)



"Dipole magnet" field (two poles)

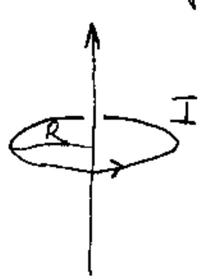
.. and similarly a quadrupole (used extensively in plasma physics, accelerators):



"Quadrupole magnet" field (four poles)

Even if it's well hidden, the field lines always loop back and close!

Force and torque on a magnetic dipole: Easiest to start with a loop of current about the z axis:



$$\vec{F} = I \oint d\vec{l} \times \vec{B}$$

$$= I\left(\oint d\vec{l}\right) \times \vec{B} \quad \text{if } \vec{B} = \text{constant}$$

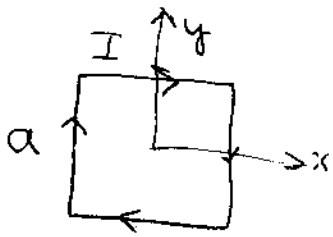
$$= 0$$

$\rightarrow$ No net force if field is constant.

If field is <u>not</u> constant, there is a net force (as with electric dipole: $\vec{F} = \text{grad}\left(\vec{m} \cdot \vec{B}(\vec{r})\right)$

(assuming the dipole is ideal — in practice, this just means current loop is small enough that second derivative of B is not important.)

Torque: Easiest to work with a square dipole:



Dipole moment $\vec{m} = Ia^2\hat{z}$

Force on left leg $= Ia\hat{y}\times\vec{B} \Longrightarrow$ Torque $\vec{N} = \frac{a}{2}(-\hat{x})\times\left(Ia\hat{y}\times\vec{B}\right)$

right leg $= Ia(-\hat{y}\times\vec{B}) \Longrightarrow$ Torque $\vec{N} = \frac{a}{2}(\hat{x})\times\left(-Ia\hat{y}\times\vec{B}\right)$

So net torque from these legs $= -Ia^2\hat{x}\times(\hat{y}\times\vec{B})$

$$= -Ia^2\left[\hat{y}(\hat{x}\cdot\vec{B}) - \vec{B}(\underbrace{\hat{x}\cdot\hat{y}}_{0})\right] = -Ia^2 B_x\hat{y}$$

From other two legs, $\vec{N} = +Ia^2\hat{y}\times(\hat{x}\times\vec{B}) = +Ia^2 B_y\hat{x}$

So total torque $\vec{N} = Ia^2\left(B_y\hat{x} - B_x\hat{y}\right)$

$= Ia^2\,\hat{z}\times\vec{B}$ But $Ia^2\hat{z} = \vec{m}$

So in general, $\vec{N} = \vec{m}\times\vec{B}$.