Vector potential:

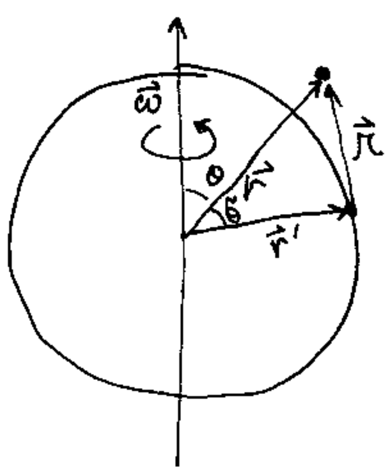
$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r}')}{\mathscr{r}} \longleftarrow$$

Three separate integrals only if $\vec{J}, \vec{A}$ are in cartesian components.

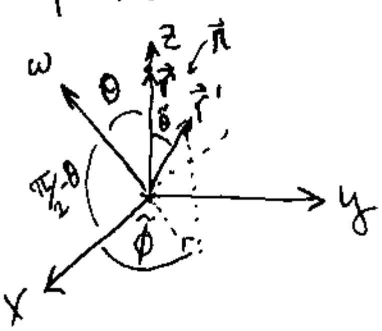
Like $V = \int d^3r' \frac{\rho}{\mathscr{r}}$, this integral only works if $\vec{J}(\vec{r}) \to 0$ as $r \to \infty$.

Example from Griffiths: a shell, radius R, of constant surface charge density σ, spinning at angular velocity $\vec{\omega}$. Find $\vec{A}(\vec{r})$.

Start with a point outside:



A convenient choice of axes: Place $\vec{r}$ along $\hat{z}$ axis, $\vec{\omega}$ in the x-z plane:



Surface current $\vec{K} = \sigma\vec{v} = \sigma\vec{\omega}\times\vec{r}'$

Now: $\vec{r} = r\hat{z}$, $\vec{r}' = R\left(\sin\tilde{\theta}\cos\tilde{\phi}\,\hat{x} + \sin\tilde{\theta}\sin\tilde{\phi}\,\hat{y} + \cos\tilde{\theta}\,\hat{z}\right)$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = -\left[R\sin\tilde{\theta}\cos\tilde{\phi}\,\hat{x} + R\sin\tilde{\theta}\sin\tilde{\phi}\,\hat{y} + (R\cos\tilde{\theta} - r)\,\hat{z}\right]$$

$$\sqrt{R^2 + r^2 - 2Rr\cos\tilde{\theta}} = \mathscr{r}$$

Now, express $\vec{K}$ in these coords: $\vec{K} = \sigma\left(\vec{\omega}\times\vec{r}'\right)$

$$\vec{\omega} = \omega\left(\cos\theta\,\hat{z} + \sin\theta\,\hat{x}\right) \quad \text{so}$$

$$\vec{K} = R\omega\left[\hat{x}\left(-\cos\theta\sin\tilde{\theta}\,\underset{\sin}{\cancel{\cos}}\,\tilde{\phi}\right) + \hat{y}\left(\cos\theta\sin\tilde{\theta}\cos\tilde{\phi} - \sin\theta\cos\tilde{\theta}\right) + \hat{z}\left(\sin\theta\sin\tilde{\theta}\sin\tilde{\phi}\right)\right]$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int dA\,\frac{\vec{K}}{r}$$

$$= \frac{\mu_0}{4\pi}\int_0^{\pi} d\tilde{\theta}\int_0^{2\pi} d\tilde{\phi}\; R^2\sin\tilde{\theta}\;\frac{\vec{K}}{r}$$

Note that x and z components integrate to zero in ϕ', so only include y component

$$\vec{A} = \frac{R^3\sigma\mu_0\omega}{4\pi}\int_0^{\pi} d\tilde{\theta}\int_0^{2\pi} d\tilde{\phi}\;\frac{\cos\theta\sin^2\tilde{\theta}\cos\tilde{\phi} - \sin\theta\cos\tilde{\theta}\sin\tilde{\theta}}{\sqrt{R^2+r^2-2Rr\cos\tilde{\theta}}}\;\hat{y}$$

$\cos\tilde{\phi}$ term integrates to zero, so

$$\vec{A} = \hat{y}\left(\frac{R^3\sigma\mu_0\omega\sin\theta}{2}\right)\int_0^{\pi} d\tilde{\theta}\;\frac{\cos\tilde{\theta}\sin\tilde{\theta}}{\sqrt{R^2+r^2-2Rr\cos\tilde{\theta}}}$$

Integral evaluates to:

$$-\frac{1}{3R^2r^2}\left[\left(R^2+r^2+Rr\right)|R-r| - \left(R^2+r^2-Rr\right)(R+r)\right]$$

$$= \frac{2}{3}\frac{r}{R^2} \text{ for } r<R, \quad = \frac{2}{3}\frac{R}{r^2} \text{ for } r>R$$

So, $\vec{A} = \hat{y}\left(\frac{R^3\sigma\mu_0\omega\sin\theta}{2}\right)\left(\frac{2}{3}\frac{r}{R^2}\right)$ inside sphere

$= \hat{y}\,\frac{1}{3}R\sigma\mu_0\,\omega r\sin\theta$ But $\vec{\omega}\times\vec{r} = \omega r\hat{y}\sin\theta$

so $\vec{A} = \frac{1}{3}R\sigma\mu_0\,\vec{\omega}\times\vec{r}$

Outside sphere: $\vec{A} = \hat{y}\left(\frac{R^3\sigma\mu_0\omega\sin\theta}{2}\right)\left(\frac{2}{3}\frac{R}{r^2}\right)$

$= \hat{y}\,\frac{1}{3}R^4\sigma\mu_0\omega r^{-2}\sin\theta$

$= \frac{1}{3}R\sigma\mu_0\left(\frac{R}{r}\right)^3\vec{\omega}\times\vec{r}$

In original coordinates, $\vec{\omega}\times\vec{r} = \omega r\sin\theta\,\hat{\phi}$, since $\vec{\omega}$ is in $\hat{z}$ direction

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What is $\vec{B}$?

$\vec{B}_{in} = \frac{1}{3}R\sigma\mu_0\,\mathrm{curl}\,\vec{\omega}\times\vec{r} = \frac{\mu_0}{3}R\sigma\omega\,\mathrm{curl}\left(r\sin\theta\hat{\phi}\right)$

$= \frac{\mu_0}{3}R\sigma\omega\left\{\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}r\sin^2\theta\right]\hat{r} + \frac{1}{r}\left[-\frac{\partial}{\partial r}\left(r^2\sin\theta\right)\right]\hat{\theta}\right\}$

$= \frac{\mu_0}{3}R\sigma\omega\left(2\cos\theta\,\hat{r} - 2\sin\theta\,\hat{\theta}\right) = \frac{2}{3}\mu_0\sigma R\omega\hat{z}$

constant field

What about B_{out}?

$$\vec{B}_{out} = \frac{\mu_0}{3} R^4 \sigma\omega \left\{ \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta} r^{-2} \sin^2\theta\right]\hat{r} + \frac{1}{r}\left[-\frac{\partial}{\partial r}\left(r^{-1}\sin\theta\right)\right]\hat{\theta} \right\}$$

$$= \frac{\mu_0}{3} R^4 \sigma\omega \frac{1}{r} \left\{ \frac{1}{r^2} 2\cos\theta\, \hat{r} + \frac{1}{r^2}\sin\theta\, \hat{\theta} \right\}$$

$$= \frac{\mu_0 R^4 \sigma\omega}{3r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

which is a perfect dipole.