Another really neat Ampère's Law example: toroid. Read about it in Griffiths!

Magnetic potential:

Electrostatics: $\vec{E} = -\text{grad}\, V$ where V=scalar.
Said this worked (or at least consistent) because $\text{curl}\, \vec{E} = 0$, and curl of a gradient is zero.

Magnetostatics: $\text{div}\, \vec{B} = 0$, know that divergence of a curl vanishes. Tempting to say that $\vec{B} = \text{curl}\, \vec{A}$. Call $\vec{A}$ the magnetic vector potential.

Now, apply Ampère's Law in terms of $\vec{A}$:

$$\mu_0 \vec{J} = \text{curl}\, \vec{B} = \text{curl}(\text{curl}(\vec{A}))$$

$$\mu_0 \vec{J} = \text{grad}(\text{div}\, \vec{A}) - \nabla^2 \vec{A} \quad \text{(inside cover of Griffiths)}$$

What is $\text{div}\, \vec{A}$? Remember curl of divergence is 0.
So, take a field $\vec{B}$ and its vector potential $\vec{A}$, and subtract $\vec{A}'$ such that $\text{div}\, \vec{A}' = +\text{div}\, \vec{A}$ and $\text{curl}\, \vec{A}'$ is zero.

→ Ambiguity of $\vec{A}$

allows one to add the gradient of any scalar field $\vec{\nabla}\lambda$: $\vec{B} = \text{curl}(\vec{A} + \text{grad}\, \lambda) = \text{curl}\, \vec{A} + \underbrace{\text{curl}\, \text{grad}\, \lambda}_{0}$

Adding $\vec{\nabla}\lambda$ to $\vec{A}$ is called a <u>gauge transformation</u>, and $\vec{B}$ is unaffected.

The gauge where $\operatorname{div}\vec{A}=0$ is called Coulomb gauge, and has nice property:

$$\mu_0\vec{J} = \operatorname{grad}\left(\cancel{\operatorname{div}\vec{A}}\right)_{\to 0} - \nabla^2\vec{A} \qquad \text{(A restatement of Ampere's Law)}$$

$$\Rightarrow \nabla^2\vec{A} = -\mu_0\vec{J}$$ which is Poisson's eqn. for a <u>vector</u>.

Recalling $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \, \frac{\rho(\vec{r}')}{\mathscr{r}}$ is solution to $\nabla^2 V = \frac{-1}{\varepsilon_0}\rho$,

by same Poisson eqn. solution:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}')}{\mathscr{r}}$$ is solution to $\nabla^2\vec{A} = -\mu_0\vec{J}$

We can actually get this from the Biot-Savart integral:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

and the identity $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = -\operatorname{grad}\frac{1}{\mathscr{r}}$ where grad is gradient with respect to *unprimed* coordinates.

and $\operatorname{curl}\left[a(\vec{r})\vec{V}(\vec{r})\right] = a(\vec{r})\operatorname{curl}\vec{V} - \vec{V}\times\operatorname{grad}a(\vec{r})$

so $$\operatorname{curl}\left(\frac{1}{\mathscr{r}}\vec{J}(\vec{r}')\right) = \frac{1}{\mathscr{r}}\underbrace{\operatorname{curl}\vec{J}(\vec{r}')}_{0} - \vec{J}(\vec{r}')\times\operatorname{grad}\frac{1}{\mathscr{r}}$$

0 because curl refers to <u>unprimed</u> $\vec{r}$.

so $\mathrm{curl}\left(\frac{1}{\mathscr{r}}\vec{J}(\vec{r}')\right) = -\vec{J}(\vec{r}') \times \mathrm{grad}\,\frac{1}{\mathscr{r}}$

$$\vec{B}(\vec{r}) = \frac{-\mu_0}{4\pi}\int d^3r'\,\vec{J}(\vec{r}') \times \mathrm{grad}\frac{1}{\mathscr{r}}$$

$$= \frac{+\mu_0}{4\pi}\int d^3r'\,\underbrace{\overrightarrow{\mathrm{grad}} \times \frac{\vec{J}(\vec{r}')}{\mathscr{r}}}_{\text{This is a curl}}$$

can move grad to left of $\vec{J}(\vec{r}')$ since only differentiates $\vec{r}$.
Can also move it outside $\vec{r}'$ integral.

$$\vec{B}(\vec{r}) = \mathrm{curl}\left[\frac{\mu_0}{4\pi}\int d^3r'\,\frac{\vec{J}(\vec{r}')}{\mathscr{r}}\right]$$

so $[\ ]$ is $\vec{A}$.

This integral form for $\vec{A}$ is usually easier to evaluate than the full Biot-Savart integral.

Finding $\vec{A}$ from a current distribution looks (componentwise) just like finding V from a static charge distribution!
<u>WARNING</u>: Can only separate components in <u>cartesian</u> coordinates. In curvilinear coordinates, the unit vectors can't come out of the integral since they're position-dependent.

$\vec{A}$ from line, surface currents:

$$\vec{A} = \frac{\mu_0}{4\pi}\int dl'\,\frac{\vec{I}}{\mathscr{r}} = \frac{I\mu_0}{4\pi}\int d\vec{l}'\,\frac{1}{\mathscr{r}}$$

path current

$$\vec{A} = \frac{\mu_0}{4\pi}\int dA'\,\frac{\vec{K}}{\mathscr{r}}$$

surface current