Example of calculating divergence: $\vec{V} = \frac{1}{\sqrt{x^2+y^2}}\hat{\rho}$, where $\hat{\rho}$ is the cylindrical radial unit vector ($\hat{s}$ in Griffiths but nowhere else on planet Earth). What is div $\vec{V}$?

$$\hat{\rho} = \frac{1}{\rho}\vec{\rho} = \frac{1}{\rho}(x\hat{x}+y\hat{y}) = \frac{1}{\sqrt{x^2+y^2}}(x\hat{x}+y\hat{y})$$

$$\longrightarrow \vec{V} = \frac{1}{\sqrt{x^2+y^2}}\frac{1}{\sqrt{x^2+y^2}}(x\hat{x}+y\hat{y}) = \frac{1}{x^2+y^2}(x\hat{x}+y\hat{y})$$

$$\text{div}\vec{V} = \vec{\nabla}\cdot\vec{V} = \frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2}$$

$$= \left(x\frac{\partial}{\partial x}\frac{1}{x^2+y^2} + \frac{1}{x^2+y^2}\frac{\partial x}{\partial x}\right) + \left(y\frac{\partial}{\partial y}\frac{1}{x^2+y^2} + \frac{1}{x^2+y^2}\frac{\partial y}{\partial y}\right) \leftarrow \text{by product rule}$$

$$= \frac{2}{x^2+y^2} + \left[x\frac{-1}{(x^2+y^2)^2}(2x) + y\frac{-1}{(x^2+y^2)^2}(2y)\right]$$

$$= \frac{2}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} - \frac{2y^2}{(x^2+y^2)^2} = \frac{2}{x^2+y^2} - 2\frac{(x^2+y^2)}{(x^2+y^2)^2}$$

$$= 0.$$

Divergence is "scalar derivative" of vector field. Analogous vector derivative is <u>curl</u>:

$$\text{curl}\,\vec{V} = \vec{\nabla}\times\vec{V} = \hat{x}\frac{\partial V_z}{\partial y} + \text{cyc. perm} - \text{anticyc. perm.}$$

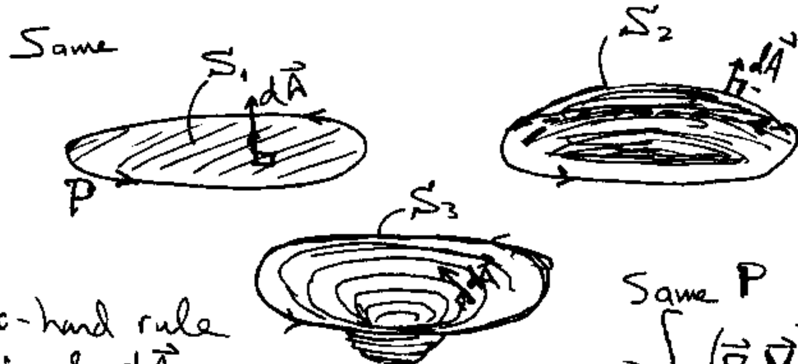
Curl can be visualized as the "swirliness" of the vector field.

Stokes's theorem:

Take a vector field and an arbitrary closed path P in space that forms the edge of a surface S:

$$\int_S (\mathrm{curl}\,\vec{V}(r)) \cdot d\vec{A} = \oint_P \vec{V} \cdot d\vec{l} .$$

Another amazing result! The path integral around the edge of a surface is equal to the integral of the curl across the surface! Also means that the integral of the curl across any two surfaces is identical if they share a boundary:



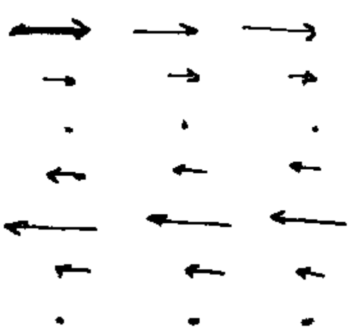
Sign convention: right-hand rule relates path direction & $d\vec{A}$.

Same P

$$\Longrightarrow \int_{S_1} (\vec{\nabla} \times \vec{V}) \cdot d\vec{A} = \int_{S_2} (\vec{\nabla} \times \vec{V}) \cdot d\vec{A} = \int_{S_3} (\vec{\nabla} \times \vec{V}) \cdot d\vec{A}$$

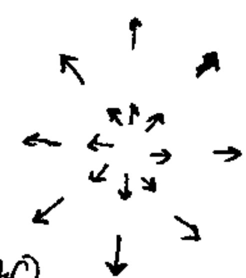
Electrostatics: curl is pretty irrelevant.

Magnetostatics: $\mathrm{curl}\,\vec{B} \leftrightarrow \vec{J}$: curl of magnetic field is the current density.



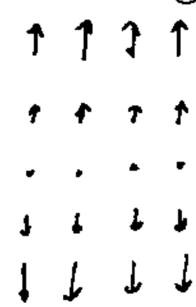
curl $\otimes$

curl $\odot$



curl=0

curl=0



Spherical coordinates

$\hat{r}, \hat{\theta}, \hat{\phi}$ form a coordinate system, but their directions change with position! So can't add components of two vectors in sph. coords. if they are defined at different positions! Rarely a problem in vector field theories in physics since vectors are fn. of position and it's uncommon to do "non-local" addition.

Volume integration: $$d^3r = (dr)(r\,d\theta)(r\sin\theta\,d\phi) = r^2\sin\theta\,dr\,d\theta\,d\phi$$

Path integration: $d\vec{l} = \hat{r}\,dr + r\,\hat{\theta}\,d\theta + r\sin\theta\,\hat{\phi}\,d\phi$

div, grad, curl: see Griffiths p. 42

Cylindrical coords: $$d^3r = (d\rho)(\rho\,d\phi)(dz) = \rho\,d\rho\,d\phi\,dz$$

div, grad, curl: see Griffiths p. 44

Curvilinear coords: not a topic where there's much to discuss, but <u>be careful!</u>