Biot-Savart law: field of a static current:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

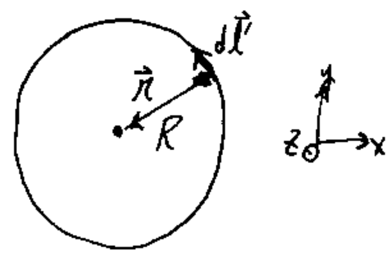
(Direct result of curl $\vec{B} = \mu_0 \vec{J}$.)

For a path current, this reduces to

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

where $d\vec{l}'$ is the path element along the current path (wire). Biot-Savart integrals can be very ugly unless some symmetry is there to help you.

Start with simple example: a loop of current $I$, radius $R$. What is $\vec{B}$ in center of loop?




$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$\mathscr{r}^2 = R^2$, and $d\vec{l}' = dl\,\hat{\phi}'$, and $\hat{\mathscr{r}} = -\hat{r}'$ so $d\vec{l}' \times \hat{\mathscr{r}} = +\hat{z}$

$$\vec{B} = \frac{\mu_0 I}{4\pi R^2} \int dl = \frac{\mu_0 I}{2R}$$

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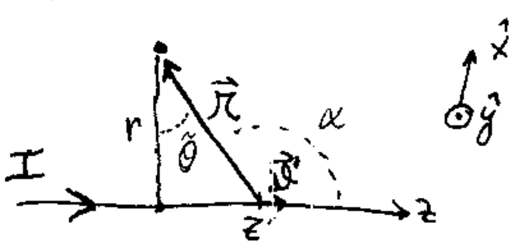
Another example: field of a wire on the z axis:



$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$d\vec{l}' \times \hat{\mathscr{r}} = dz' \sin\alpha \; \hat{y}$$
$$= dz' \cos\tilde{\theta} \; \hat{y}$$

$$\pi - \alpha = \frac{\pi}{2} - \tilde{\theta}$$

so $\tilde{\theta} = \frac{\pi}{2} - \alpha$

Now, find $dz'$ in terms of $\tilde{\theta}$:

$$z' = r \tan\tilde{\theta}$$

so $$dz' = \frac{r\,d\tilde{\theta}}{\cos^2\tilde{\theta}}$$

$$\Rightarrow d\vec{l}' \times \hat{\mathscr{r}} = \frac{r\,d\tilde{\theta}}{\cos\tilde{\theta}}\,\hat{y}$$

Now, consider $\frac{1}{\mathscr{r}^2} = \frac{\cos^2\tilde{\theta}}{r^2}$.

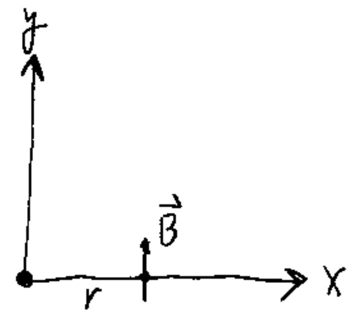
$$\vec{B} = \frac{\mu_0 I \hat{y}}{4\pi}\int\left(\frac{r\,d\tilde{\theta}}{\cos\tilde{\theta}}\right)\left(\frac{\cos^2\tilde{\theta}}{r^2}\right) = \frac{\mu_0 I \hat{y}}{4\pi r}\int d\tilde{\theta}\cos\tilde{\theta}$$

Integration is for complete line: $z'$ from $-\infty$ to $\infty$

so $\tilde{\theta}$ goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$:

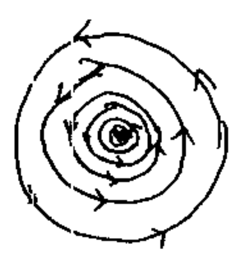
$$\vec{B} = \frac{\mu_0 I \hat{y}}{2\pi r}.$$

Looking along wire:




$\vec{B}$ is always azimuthal, falls off as $\frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}}$.

Field lines:



in any location, $\vec{B} = \frac{\mu_0 I \hat{\phi}}{2\pi r}$

Check vs. definition of $\vec{B}$:

$\oint d\vec{l}\cdot\vec{B}$ along circle of radius r:

$$= \left(\frac{\mu_0 I \hat{\phi}}{2\pi r}\right)\cdot\left(2\pi r\hat{\phi}\right) = \mu_0 I$$

Now, compare to original statement that $\text{curl}\,\vec{B} = \mu_0 \vec{J}$:

Stokes's theorem says $\oint d\vec{l}\cdot\vec{B} = \int_S d\vec{A}\cdot \text{curl}\,\vec{B}$

$$= \mu_0 \int_S d\vec{A}\cdot\vec{J}$$

$$= \mu_0 I_{\substack{\text{through}\\ \text{surface}}} \quad \checkmark$$