P3310 LECTURE 27

Magnetostatics

EXAM: ROOM FA N141; ALERT ME RE: CONFLICTS;
MONDAY 4/2: LECTURE → TOPICAL REVIEW
EVENING → Q&A

→ Physics of charges and currents in situations where current does not change with time.

Need to introduce magnetic field: $\vec{B}$ has dimensions of tesla (SI) / gauss (gaussian). $1\,T = 10^4\,G$

Fundamental relations in vacuum magnetostatics:

Force law: $\vec{F} = q(\vec{v} \times \vec{B})$ $\vec{v}$ is velocity

Field origin: $\text{curl}\,\vec{B} = \mu_0 \vec{J}$ $\vec{J}$ is current density

$\text{div}\,\vec{B} = 0$

...where $\mu_0 \equiv 4\pi \times 10^{-7} \frac{N}{A^2}$ $= \frac{kg \cdot m}{C^2}$ "permeability of free space"

Once again, this is not a physically meaningful property of space, just an artifact resulting from definition of Coulombs.

Gaussian unit equivalents: $\vec{F} = q\left(\frac{\vec{v}}{c} \times \vec{B}\right)$ where c = speed of light

$$\text{curl}\,\vec{B} = \frac{4\pi}{c}\vec{J}$$

$$\text{div}\,\vec{B} = 0$$

→ Note gaussian symmetry: adding $\vec{E}$, $\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$

$\frac{\vec{v}}{c}$ is dimensionless (nice!) (and very physical), so $\vec{E}$ and $\vec{B}$ have same dimensions!

Immediate application of the Lorentz force law:

$\vec{F} = q\vec{v}\times\vec{B}$ Note cross-product with velocity: $\vec{F} \perp \vec{v}$ so $\vec{F}_{mag}$ can do no work. (This is a direct result of the empirical law $\mathrm{div}\vec{B}=0$:) if there were magnetic charges, then $\vec{B}$ could do work on them.

Motion of charge in a uniform $\vec{B}$ field: $\vec{F} \perp \vec{v}$, with $|v|$ and $\vec{B}$ constant $\Rightarrow$ centripetal force! Circular motion.

$$qvB = \frac{mv^2}{R} \quad \Rightarrow \quad R = \frac{mv}{qB} = \frac{p}{qB}$$

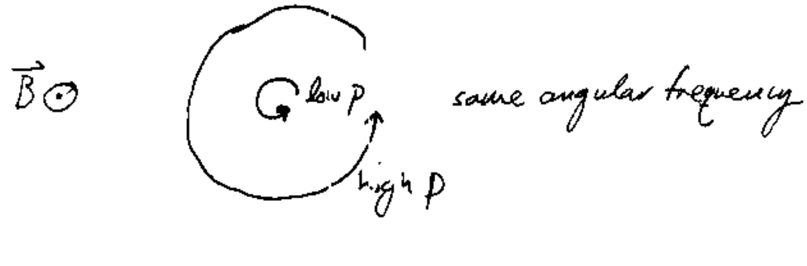
Used by particle physicists routinely to measure momentum of charged particles: let them enter a $\vec{B}$ field, then track their motion and measure curvature radius.

(Allowed path is actually helical, since any component of momentum parallel to $\vec{B}$ is not affected!)

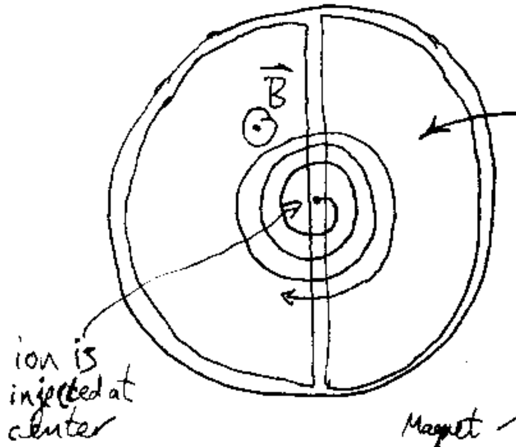
Generally called cyclotron motion. Angular frequency is

$\omega = \frac{v_\perp}{R} = \frac{qB}{m}$ Note -- doesn't depend on v or R.

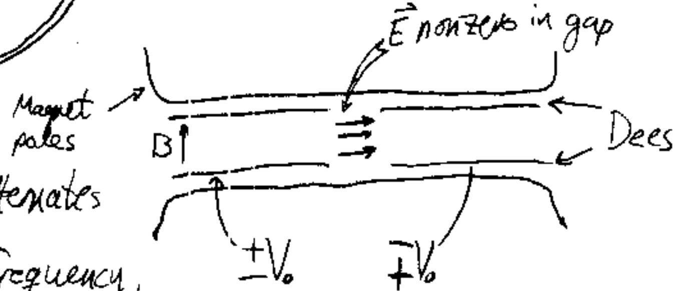
A nice consequence: particles of any speed (energy) will rotate at same frequency (but different radius) in a constant field:



Principle of cyclotron (early particle accelerator):



Circular region of constant $\vec{B}$
Two pairs of D-shaped electrode plates above and below, with alternating voltage applied to the two D's:

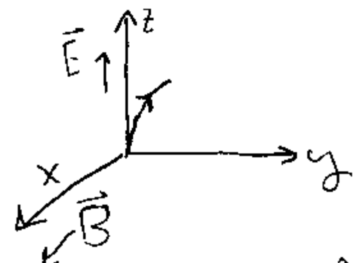


Voltage on the D's alternates at the cyclotron frequency,
so particle that sees $\Delta V = 2V_0$ gap at $\phi = 0$ will see $-2V_0$ gap at $\phi = \pi$.
$\rightarrow$ particle thus sees clockwise force at both gap crossings. At each crossing, work done is $2QV_0$ so in n orbits it gains $4nQV_0$ energy.

Acceleration continues until the particle reaches the edge of the magnetic field region. Technology is limited in energy available by the ability to build large magnets, and then by relativistic effects that affect cyclotron motion formula. Typical energies available are ~200 MV·Q.

Cycloid motion: uniform, perpendicular $\vec{B}$ and $\vec{E}$:
$\vec{B} = B\hat{x}$, $\vec{E} = E\hat{z}$:

Particle starts at $(0,0,0)$ at rest. $\vec{F} = Q(\vec{E} + \vec{v}\times\vec{B})$



$= QE\hat{z}$ initially, but soon obtains a force in y-z plane.

No force can exist in $\hat{x}$.

$$\vec{F} = Q\left(E\hat{z} + B\dot{z}\hat{y} - B\dot{y}\hat{z}\right) = m\vec{a} = m\left(\ddot{y}\hat{y} + \ddot{z}\hat{z}\right)$$

By components: coupled differential eqns:

$$QB\dot{z} = m\ddot{y}, \quad Q\left(E - B\dot{y}\right) = m\ddot{z}.$$

Solution in general is

$$y(t) = C_1 \cos\omega t + C_2 \sin\omega t + \frac{E}{B}t + C_3$$
$$z(t) = C_2 \cos\omega t - C_1 \sin\omega t + C_4$$

where $\omega = \frac{QB}{m}$

4 constraints define the constants: $y(0) = z(0) = \dot{y}(0) = \dot{z}(0) = 0$.

$$y(t) = \underbrace{\frac{E}{\omega B}\left(\omega t - \sin\omega t\right)}_{\text{grows; oscillating offset}} \qquad z(t) = \underbrace{\frac{E}{\omega t}\left(1 - \cos\omega t\right)}_{\text{oscillates only.}}$$

Define $R \equiv \frac{E}{\omega B} = \frac{mE}{QB^2}$

$$y - R\omega t = -\sin\omega t \qquad z - R = \cos\omega t$$

square both eqns:

$$\left(y - R\omega t\right)^2 = \sin^2\omega t \qquad \left(z - R\right)^2 = \cos^2\omega t$$

so $\left(y - R\omega t\right)^2 + \left(z - R\right)^2 = R^2$

Which is eqn. of cycloid motion, with speed $v = \omega R = \frac{E}{B}$.