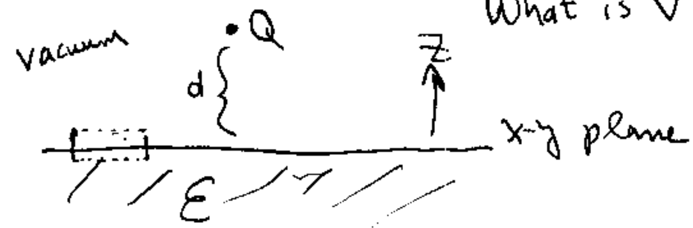
Another dielectric example: charge above a thick dielectric:
What is V everywhere?



$z<0$: dielectric w/ constant $\frac{\varepsilon}{\varepsilon_0} = K$.

Will treat this somewhat differently vs. book:
Laplace eqn. applies everywhere except $(x,y,0)$ and $(0,0,d)$.
- $V \to 0$ at $r \to \infty$. - Bound charge should only be on surface. (why?)
- At boundary: Gaussian pillbox, small area: $\oint d\vec{A} \cdot \vec{D} = Q_{free} = 0$

$$D_{z\,up} = -D_{z\,dn} \quad \text{(no free charge)}$$

so:
$$\varepsilon_0 \frac{\partial V_{up}}{\partial z}\bigg|_{z=0} = \varepsilon \frac{\partial V_{dn}}{\partial z}\bigg|_{z=0}$$

Now, let's suppose an image charge Q_i at $(0,0,-d)$. What value of Q_i gives V_{up} that satisfies the boundary conditions?
→ Need V_{dn} too! It's not zero as in conductor case.

Think about what an image charge is: The σ_b at surface is arranged to appear from above like the field of the point image charge. But that's symmetric — so it appears from below as the same point charge, at an equivalent distance above! Therefore:

$$V_{up} = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{Q_i}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

$$V_{dn} = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q+Q_i}{\sqrt{x^2+y^2+(z-d)^2}}\right]$$

(Note consistency with conductor case, where $Q_i = -Q \Rightarrow V_{dn} = 0$)

$$\frac{\partial V_{up}}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{-Q(z-d)}{[x^2+y^2+(z-d)^2]^{\frac{3}{2}}} + \frac{-Q_i(z+d)}{[x^2+y^2+(z+d)^2]^{\frac{3}{2}}}\right\}$$

$$\frac{\partial V_{dn}}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{-(Q+Q_i)(z-d)}{[x^2+y^2+(z-d)^2]^{\frac{3}{2}}}\right\}$$

At $z=0$:

$$\varepsilon_0\frac{\partial V_{up}}{\partial z} = \frac{d}{4\pi\ (x^2+y^2+d^2)^{\frac{3}{2}}}\left(Q-Q_i\right)$$

$$\varepsilon\frac{\partial V_{dn}}{\partial z} = \frac{\varepsilon d}{4\pi\varepsilon_0(x^2+y^2+d^2)^{\frac{3}{2}}}\left(Q+Q_i\right)$$

set equal, solve for Q_i

$$\frac{\varepsilon}{\varepsilon_0}(Q+Q_i) = Q-Q_i$$

$$KQ+KQ_i = Q-Q_i$$

$$(K+1)Q_i = -Q(K-1)$$

$$Q_i = -Q\frac{K-1}{K+1} = -Q\frac{\chi_e}{\chi_e+2}$$

From here, can calculate σ_b easily.

Force & Energy in Dielectrics

Vacuum fields: found $W = \frac{\varepsilon_0}{2}\int d^3r\, E^2(r)$; for capacitor $= \frac{1}{2}CV^2$

Charge up a capacitor, and the charges can't tell if there's a dielectric or just geometry causing the potential V. So, $W=\frac{1}{2}CV^2$ must apply even if dielectric present.

If $C_{diel.} = K\, C_{vac}$, then:

$W_{diel} = K W_{vac}$ (for same V). $\rightarrow$ But $\vec{E}_{diel} = \vec{E}_{vac}$ for same V (of course Q is different)

$$\text{So } \frac{\varepsilon_0}{2}\int d^3r\, E^2 \Longrightarrow \frac{\varepsilon_0}{2}\int d^3r\, K E^2$$

$$KE^2 = K\vec{E}\cdot\vec{E} = \frac{\varepsilon}{\varepsilon_0}\vec{E}\cdot\vec{E} = \frac{1}{\varepsilon_0}\vec{D}\cdot\vec{E} \Rightarrow W = \frac{1}{2}\int d^3r\, \vec{D}\cdot\vec{E}$$

Apparent conflict: $\vec{D}\cdot\vec{E} \neq \varepsilon_0 E^2$?

Deriving $\vec{D}\cdot\vec{E}$, explicitly ignored any work done on the bound charge. If we include that, then E^2 is the "right" answer. Griffiths points out, though, that there's also a potential energy associated with deforming the dielectric molecules to create polarization. (A bit awkward, though, as a concept!) Since the lowest-energy state is clearly zero field, you can't extract energy from a dielectric, so $W_{pot.} = -W_{bound}$ and $W_{tot} = W_{free}$.

Now, look closer. Remember $\frac{\varepsilon_0}{2}E^2$ is the energy density of the field. In vacuum, interpretation is easy. In a dielectric, $\vec{E}$ is the <u>macroscopic</u> (average) field. That's fine for calculating anything involving Q_{free}, or $V = -\int d\vec{l}\cdot\vec{E}$, since microscopic fluctuations average to zero. But, if you are integrating E^2, the microscopic fields should matter!