P3310 LECTURE 24

... calculation continued from last time:

$$(*) \qquad A_\ell = B_\ell R^{-(2\ell+1)} \qquad \ell \neq 1$$
$$(**) \qquad A_1 = B_1 R^{-3} - E_0 \qquad \ell = 1$$

Now, use derivative condition:

$$\varepsilon \frac{\partial V_{in}}{\partial r} = \varepsilon_0 \frac{\partial V_{out}}{\partial r} \quad \text{at } r=R$$

$$\varepsilon \sum_\ell A_\ell \ell r^{\ell-1} P_\ell(\cos\theta)\Big|_{r=R} = \varepsilon_0 \left[-E_0\cos\theta + \sum_\ell B_\ell(-\ell-1) r^{-(\ell+2)} P_\ell(\cos\theta)\right]\Bigg|_{r=R}$$

$$\varepsilon \sum_\ell A_\ell \ell R^{\ell-1} P_\ell(\cos\theta) = -\varepsilon_0 \sum_\ell \left[B_\ell(\ell+1)R^{-(\ell+2)} + E_0\delta_{\ell 1}\right] P_\ell(\cos\theta)$$

Now, exploit orthogonality again and match up individual $P_\ell$ coefficients:

$$\varepsilon A_\ell \ell R^{\ell-1} = -\varepsilon_0\left[B_\ell(\ell+1)R^{-(\ell+2)} + E_0\delta_{\ell 1}\right]$$

<u>$\ell \neq 1$:</u> $\varepsilon A_\ell = -\varepsilon_0 B_\ell \frac{\ell+1}{\ell} R^{-(2\ell+1)}$

so $A_\ell = \frac{-\varepsilon_0}{\varepsilon} \frac{\ell+1}{\ell} R^{-(2\ell+1)} B_\ell$

... which can only be reconciled with $(*)$ if $A_\ell = B_\ell = 0$.

Now, look at $\ell = 1$:

$$\varepsilon A_1 = -\varepsilon_0\left[2B_1R^{-3} + E_0\right]$$

Solve simultaneously with $(**)$ to find: $A_1 = \frac{-3E_0}{K+2} \qquad B_1 = \frac{K-1}{K+2}R^3E_0$

$$V_{in} = \frac{-3E_0}{K+2} r\cos\theta = \frac{-3E_0 z}{K+2} \Rightarrow \vec{E} = \frac{+3}{K+2}\vec{E_0}.$$ <u>Constant</u> field!

$$V_{out} = -E_0 z + B_1 \cos\theta$$

$$= -E_0 z + \frac{K-1}{K-2} R^3 E_0 \cos\theta \quad \Rightarrow \text{ uniform + dipole field}$$

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Another approach: use linear polarizability directly: $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ and do a series of approximations: Let $\vec{E}_0 = E_0 \hat{z}$ be the initial field, and calculate $\vec{P}_0$ = "initial" response polarization… then $\vec{E}_1$ = field due to $\vec{P}_0$, $\vec{P}_1$ = polarization response to $E_1$, …

So $\vec{E} = \vec{E}_0 + \vec{E}_1 + \vec{E}_2 + \cdots$

Begin: $\vec{P}_0 = \varepsilon_0 \chi_e E_0 \hat{z}$

→ Now, recall calculation last week of field inside uniformly polarized sphere:

$$\vec{E}_1 = \frac{-\vec{P}_0}{3\varepsilon_0}\hat{z} = -\frac{\chi_e}{3} E_0 \quad \text{(also uniform)}.$$

Now, $\vec{P}_1 = \varepsilon_0 \chi_e \vec{E}_1 = -\varepsilon_0 \frac{\chi_e^2}{3} E_0 \hat{z}$

so $\vec{E}_2 = \frac{-\vec{P}_1}{3\varepsilon_0} = \left(\frac{-\chi_e}{3}\right)^2 E_0 \hat{z}$

Clearly $\vec{E}_n = \left(-\frac{\chi_e}{3}\right)^n \vec{E}_0$ so $\vec{E} = \left(\sum_{n=0}^{\infty} \left(\frac{-\chi_e}{3}\right)^n\right) \vec{E}_0$

$$= \left(\frac{1}{1 + \frac{\chi_e}{3}}\right) \vec{E}_0$$

…now recall $K = 1 + \chi_e$ $$= \left(\frac{3}{3 + \chi_e}\right) \vec{E}_0$$

so $$\vec{E}_{in} = \left(\frac{3}{K+2}\right) \vec{E}_0$$