P3310 LECTURE 23

Last lecture, defined displacement field:

$$\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P} \qquad \left(\text{Gaussian units: } \vec{D} = \vec{E} + 4\pi\vec{P}\right)$$

so $\operatorname{div}\vec{D} = \rho_f$, $\operatorname{curl}\vec{D} = \operatorname{curl}\vec{P}$.

In linear dielectric material, $\vec{P} = \varepsilon_0 \chi_e E$

Define $\varepsilon = \varepsilon_0(1+\chi_e)$ permittivity

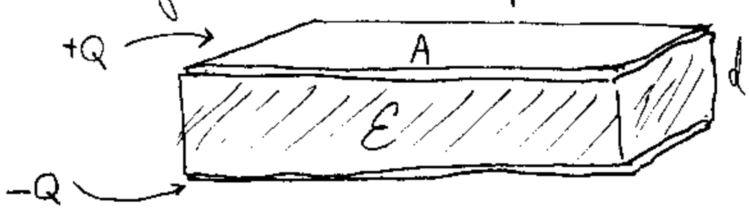
$K = \varepsilon_r = \varepsilon/\varepsilon_0$ relative permittivity (just ε in Gaussian units)

so $\vec{D} = \varepsilon\vec{E}$ ↳ often called the dielectric constant.

→ Now, we have for (linear) dielectrics a field which appears to behave like $\vec{E}$ since it's proportional... but unfortunately, ε may not (usually isn't) constant, so should write

$$\vec{D}(\vec{r}) = \varepsilon(\vec{r})\vec{E}(\vec{r})$$

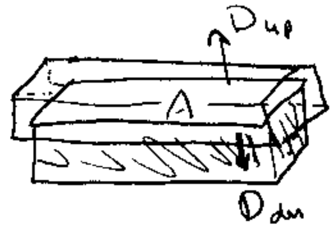
ε is a material property — ε usually changes abruptly at boundaries. Note that inside a uniform dielectric, ε constant, so $\operatorname{curl}\vec{D} = 0$ and $\vec{D}$ really does behave somewhat like $\vec{E}$. This is most useful in a parallel plate capacitor, where the entire region between the plates is filled with dielectric:



+Q and −Q are free charges

Field is contained between the plates (roughly), so consider ε constant. Now $\vec{D}$ behaves like $\vec{E}$ if material were vacuum (except for ε_0 factor):

Use Gaussian pill box: $\oint d\vec{A}\cdot\vec{D} = Q_{free}$



D_{up} $\oint d\vec{A}\cdot\vec{D} = A\underbrace{D_{up}}_{0} + AD_{down}$

$$\Rightarrow AD_{down} = Q$$

$$\vec{D} = -\frac{Q}{A}\hat{z} \quad \text{between plates}$$

So $\vec{D} = \frac{-Q}{A}\hat{z}$

But $V = -\int d\vec{l}\cdot\vec{E} = -\int d\vec{l}\cdot(\vec{D}/\varepsilon)$

$$= -\int_0^d dz\left(\frac{-Q}{A\varepsilon}\right) = +\frac{Qd}{A\varepsilon}$$

So $C = \frac{Q}{V} = \frac{A\varepsilon}{d}$ vs. $\frac{A\varepsilon_0}{d}$ for vacuum capacitor. (Nearly all commercial capacitors use dielectrics)

$\Longrightarrow$ capacitance is increased by factor of $K = \frac{\varepsilon}{\varepsilon_0}$.

Why does this work? Bound charge on top of dielectric is in contact with the conductor plate! What is σ_b?

Polarization is $\vec{P} = \varepsilon_0\chi_e\vec{E} = \varepsilon_0\chi_e\left(\frac{-Q}{A\varepsilon}\right)\hat{z}$

$$= \frac{-Q}{A}\chi_e\frac{\varepsilon_0}{\varepsilon}$$

Since $\sigma_b = \vec{P}\cdot\hat{n}$, and $\hat{n} = \hat{z}$,

$$\sigma_b = \frac{-Q}{A}\frac{\chi_e\varepsilon_0}{\varepsilon} = -\sigma_{free}\left(\frac{\chi_e}{1+\chi_e}\right)$$

So large χ: , $\sigma_b \to -\sigma_{free}$ which makes some sense since σ_b then almost cancels σ_{free}. This really does decrease the potential since the volume of high-field region is near zero. Note that there is a large force between the conductor (σ_{free}) and dielectric (σ_{bound}) —— $\vec{F}_{net}$ on dielectric is canceled by F on lower end.

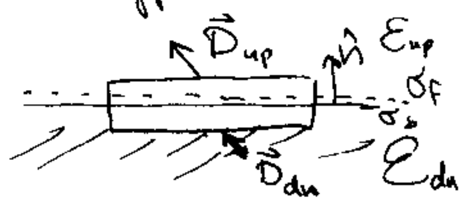
Boundary conditions in dielectrics:

<u>Inside</u> a neutral <u>uniform</u> dielectric, $\rho_b = \rho_f = 0$, so $\nabla^2 V = 0$.

(if you stick free charge inside, $\rho_b = -\operatorname{div}\vec{P} = -\operatorname{div}\left(\varepsilon_0 \frac{\chi_e}{\varepsilon}\vec{D}\right)$ $= -\varepsilon_0 \frac{\chi_e}{\varepsilon}\rho_f$

$\rho_b = -\varepsilon_0 \frac{\chi_e}{\varepsilon}\rho_f$, so $\rho_b \propto \rho_f$.)

More commonly, free charge is outside of the dielectric, so $\nabla^2 V = 0$ applies. Use Gauss's Law to find bd. condition:



$$\vec{D}_{up}\cdot\hat{n} - \vec{D}_{dn}\cdot\hat{n} = \sigma_f$$

so $\varepsilon_{up}E_{\perp up} - \varepsilon_{dn}E_{\perp dn} = \sigma_f$ but $E_\perp = -\frac{\partial V}{\partial n}$

$$\Rightarrow \varepsilon_{up}\frac{\partial V_{up}}{\partial n} - \varepsilon_{dn}\frac{\partial V_{dn}}{\partial n} = -\sigma_f$$ <u>derivative</u> boundary cond.

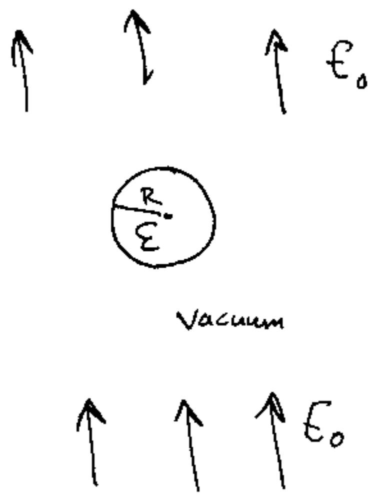
(if ε_{up} is vacuum, then great – but not assuming that.)

Also, as usual, $V_{up} = V_{dn}$ at boundary.

Example: dielectric sphere in vacuum with uniform ambient field $\vec{E}_0$.

Similar to conducting sphere, but:

- dielectric is <u>not</u> equipotential
- $\vec{E} \neq 0$ inside dielectric



Boundary conditions:

At $r=R$,

$$\varepsilon \frac{\partial V_{in}}{\partial r} = \varepsilon_0 \frac{\partial V_{out}}{\partial r} \quad (\text{since } \sigma_{free} = 0)$$

At $r \to \infty$, $V(r) = +E_0 z = +E_0 r\cos\theta$

Sep. of variables: $V(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l P_l(\cos\theta) + B_l r^{-(l+1)} P_l(\cos\theta) \right]$

inside: $B_l = 0$: $V_{in}(r,\theta) = \sum_l A_l r^l P_l(\cos\theta)$

outside: Careful! The $\vec{E}_0$ field must come from some non-accounted-for charge! So

$$V_{out}(r,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta).$$

- Match them up at $r=R$:

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = -E_0 R \underbrace{\cos\theta}_{\text{This is } P_1!} + \sum_l B_l R^{-(l+1)} P_l(\cos\theta)$$

Since solutions must be valid at all θ, this means each P_l must have matched value at R:

For $l \neq 1$: $A_l R^l = B_l R^{-(l+1)}$

$l=1$: $A_1 R = B_1 R^{-2} - E_0 R$