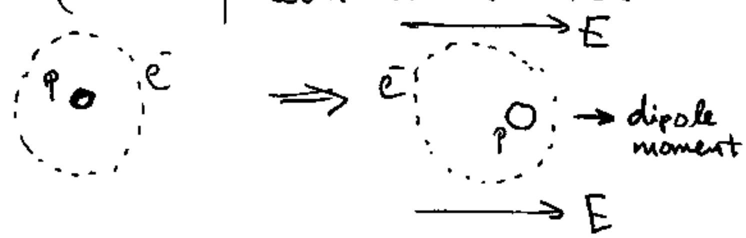
P3310 LECTURE 21

Polarized materials:

Insulating materials are generally polarizable to some extent. This means they respond to an external $\vec{E}$ by deforming atoms & molecules to (somewhat) counteract the field:

Unpolarized atom:



Atoms each acquire an induced dipole moment $\vec{p} = \alpha \vec{E}$

α = atomic polarizability

If material is not isotropic, must consider how $\vec{p}$ responds to direction of $\vec{E}$:

$$\vec{p}_i = \sum_{j=1}^{3} \alpha_{ij} E_j \quad \text{where } \alpha_{ij} \text{ is the polarizability tensor}$$

In a material, all molecules become polarized together:

$$\vec{P} = \text{dipole moment per volume} = \alpha \vec{E} \cdot n \quad \text{where } n = \frac{\text{molecules}}{\text{volume}}$$

Potential of a polarized object: start from polarization $\vec{P}$: treat as a collection of dipoles $\vec{p}$:

$$V_p(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\hat{r} \cdot \vec{p}}{r^2} \Rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V d^3r' \, \frac{\hat{\mathscr{r}} \cdot \vec{P}(\vec{r}')}{\mathscr{r}^2}$$

... where $\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}'$ as usual.

Define div' = divergence with respect to $\vec{r}'$ coordinates:

$$\text{div}'\left(\frac{1}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}^2} \hat{\mathscr{r}}$$

Identify this with integrand:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V d^3r' \; \vec{P}(\vec{r}\,') \cdot \mathrm{div}'\left(\frac{1}{\mathscr{r}}\right)$$

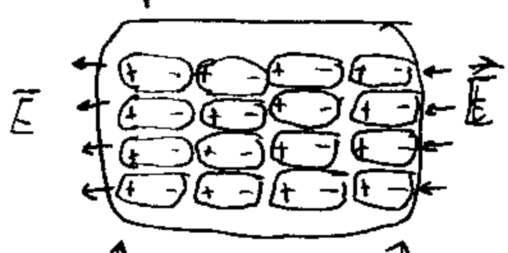
Recall: $\mathrm{div}\left[f(\vec{r})\vec{A}(\vec{r})\right] = f \,\mathrm{div}\vec{A} + \vec{A}\cdot \mathrm{grad}(f)$, and integrate by parts:
$$\frac{\vec{P}\cdot\hat{\mathscr{r}}}{\mathscr{r}^2} = \mathrm{div}'\left(\frac{\vec{P}(\vec{r}\,')}{\mathscr{r}}\right) - \frac{1}{\mathscr{r}}\mathrm{div}'\vec{P}(\vec{r}\,')$$

$$\Rightarrow V = \frac{1}{4\pi\varepsilon_0}\left[\int_V d^3r' \,\mathrm{div}'\left(\frac{\vec{P}}{\mathscr{r}}\right) - \int_V d^3r' \,\frac{\mathrm{div}'\vec{P}(\vec{r}\,')}{\mathscr{r}}\right]$$

Now, apply divergence theorem:

$$= \underbrace{\frac{1}{4\pi\varepsilon_0}\oint_S \frac{1}{\mathscr{r}}\vec{P}(\vec{r}\,')\cdot d\vec{A}'}_{\text{Looks like potential of a surface charge } \sigma_b = \vec{P}\cdot\hat{n}} - \underbrace{\frac{1}{4\pi\varepsilon_0}\int_V d^3r' \,\frac{\mathrm{div}'\vec{P}(\vec{r}\,')}{\mathscr{r}}}_{\text{Looks like potential of a volume charge } \rho_b = -\mathrm{div}\vec{P}.}$$

A physical interpretation:



Unpaired charges at edges.

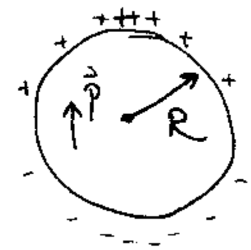
If the density of dipoles is uniform, every + end is balanced by a − end, except at edges of material.

$\Rightarrow$ if $\vec{P}(\vec{r})$ non-constant in material, this may not be true.

So $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\oint_S \frac{\sigma_b}{\mathscr{r}}\,dA' + \int_V d^3r' \,\frac{\rho_b}{\mathscr{r}}\right]$$

σ_b and ρ_b are bound charges in the material.

Field of a uniformly polarized sphere: Take $\vec{P} = P\hat{z}$.



$\text{div}\,\vec{P} = 0$ since $\vec{P}$ constant.

$\sigma_b = \vec{P}\cdot\hat{n}$ on surface

$= P\cos\theta$

So $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \, \frac{R^2 \sin\theta' \, P\cos\theta'}{\mathscr{r}}$$

... which is actually equal to (see p. 143)

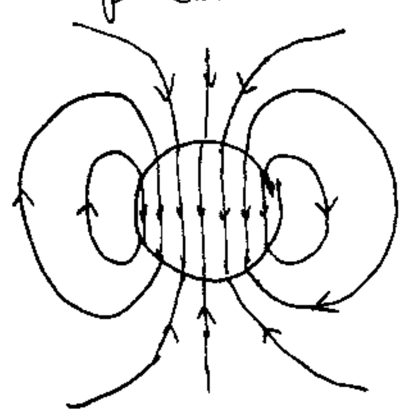
$$V(r,\theta) = \frac{Pr\cos\theta}{3\varepsilon_0} \quad (r \le R) \quad = \frac{Pz}{3\varepsilon_0}$$

$$= \frac{PR^3}{3\varepsilon_0 r^2}\cos\theta \quad r \ge R \quad = \frac{1}{4\pi\varepsilon_0}\left(\frac{4}{3}\pi R^3 P\right)\frac{\cos\theta}{r^2}$$

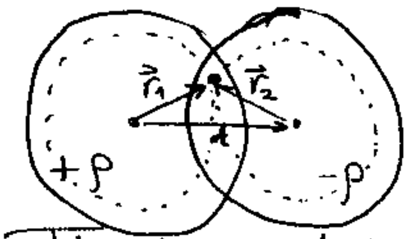
First corresponds to <u>uniform</u> field $E = -\frac{P}{3\varepsilon_0}\hat{z}$ inside

Second corresponds to pure dipole field outside, with $\vec{p} = \frac{4}{3}\pi R^3 \vec{P}$

= polarization • volume of sphere:



Another analysis of same problem: Recall problem set 3, prob. 5: (Griffiths 2.18): Field in 2 overlapping, uniformly charged spheres:



Distance vector between sphere centers is $\vec{d}$, spheres have radius R.

This is a classic superposition problem: Find field at point shown in the overlap region: $\vec{E} = \vec{E}_1 + \vec{E}_2$

$$\vec{E}_1 = \left(\frac{1}{4\pi\varepsilon_0}\frac{Q_{eff1}}{r_1^2}\right)\hat{r}_1 \qquad E_2 = \left(\frac{1}{4\pi\varepsilon_0}\frac{Q_{eff2}}{r_2^2}\right)\hat{r}_2$$

Q_{eff1} = Q from sphere 1 enclosed in $r_1 = \frac{4}{3}\pi r_1^3 \rho$

Q_{eff2} = Q from sphere 2 enclosed in $r_2 = -\frac{4}{3}\pi r_2^3 \rho$

so $\vec{E}_1 = \frac{\rho}{3\varepsilon_0} r_1 \hat{r}_1 = \frac{\rho}{3\varepsilon_0}\vec{r}_1$, $\vec{E}_2 = \frac{-\rho}{3\varepsilon_0}\vec{r}_2$

$$\Rightarrow \vec{E} = \frac{\rho}{3\varepsilon_0}(\vec{r}_1 - \vec{r}_2) = \frac{\rho}{3\varepsilon_0}\vec{d}\ .$$

But outside both spheres, field is that of 2 point charges separated by $\vec{d} \Rightarrow$ a dipole with $\vec{p} = Q\vec{d}$ where $Q = \frac{4}{3}\pi R^3 \rho$. If $d \ll R$, this looks exactly like the bound charge scenario:

- $\rightarrow$ constant $\vec{E}$ in interior, where $\rho = 0$
- $\rightarrow$ dipole field outside
- $\rightarrow$ bound charge looks like surface charge if $\vec{d}$ is very small compared to R (i.e. vast majority of spheres are in "overlap" region.