P3310 | LECTURE 20

Multipole expansion: some clarifications.

Last lecture found an expansion for $\frac{1}{\mathscr{r}} = \frac{1}{|\vec{r}-\vec{r}'|}$ in $\left(\frac{r'}{r}\right)$ and $\underline{\cos\theta'}$ (Griffiths notation). This is not the coordinate θ'! It's the angle between $\vec{r}$ and $\vec{r}'$!

Call it $\tilde{\theta}$.

Also, for this discussion of multipole expansion of a charge distribution, we are dropping our azimuthal symmetry assumption.

Expansion for $\frac{1}{\mathscr{r}}$: $\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\tilde{\theta}$.

$$= r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\tilde{\theta}\right]$$

Since r assumed large compared to any r' where $\rho(\vec{r}')\neq 0$, take $\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\tilde{\theta}\right) \equiv \epsilon$ and say it's small:

$$\mathscr{r}^2 = r^2(1+\epsilon) \Rightarrow \frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-\frac{1}{2}}$$

Binomial expansion gives:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\tilde{\theta})$$

Note r can be taken out of integral $d\vec{r}'$.

So $$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\vec{r}')}{\mathscr{r}} = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty} r^{-(n+1)}\int d^3r' \rho(\vec{r}')(r')^n P_n(\cos\tilde{\theta})$$

The first few terms:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int d^3r'\,\rho(\vec{r}') + \frac{1}{r^2}\int d^3r'\,\rho(\vec{r}')\,r'\cos\tilde{\theta} + \frac{1}{r^3}\int d^3r'\,\rho(\vec{r}')(r')^2\left(\frac{3}{2}\cos^2\tilde{\theta} - \frac{1}{2}\right) + \dots\right]$$

Looks much like previous sep. of vars expansion except:

→ No azimuthal symmetry assumption: $\vec{r}-\vec{r}'$ can point anywhere
So we can treat any $\rho(r,\theta,\phi)$ this way.

First term: just $\frac{Q}{r}$ (monopole)

Second term: dipole — but note that it's $\cos\tilde{\theta}$ not $\cos\theta'$:

Recall $\vec{r}\cdot\vec{r}' = rr'\cos\tilde{\theta}$

so $r'\cos\tilde{\theta} = \left(\frac{\vec{r}}{r}\right)\cdot\vec{r}' = \hat{r}\cdot\vec{r}'$

So dipole term is $\frac{1}{4\pi\varepsilon_0 r^2}\int d^3r'\,\rho(\vec{r}')\,\hat{r}\cdot\vec{r}'$ which seems silly

→ but it allows us to get rid of $\tilde{\theta}$ and take $\vec{r}$ completely out of integral:

$$V_{dip} = \frac{1}{4\pi\varepsilon_0 r^2}\,\hat{r}\cdot\int d^3r'\,\vec{r}'\rho(\vec{r}')$$

where $\int d^3r'\,\vec{r}'\rho(\vec{r}') \equiv \vec{p}$ is the (vector)

<u>dipole moment.</u> $V_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$ or $\frac{p_r}{4\pi\varepsilon_0 r^2}$

$\vec{p} = \sum_i q_i\vec{r}_i$ for a collection of point charges.

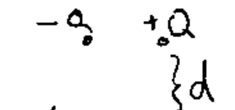
Compare to a physical dipole: $+Q$ and $-Q$ distance d apart, on $\pm z$ axis:

$$\vec{p} = Q\frac{d}{2} + (-Q)\frac{-d}{2} = Qd.$$

... which has a "perfect" dipole potential when $d \to 0$ and $Q \to \frac{|P|}{d}$.

→ The dipole can point in any direction, not just $\hat{z}$ (ie. there can be ϕ dependence).

What about quadrupole moment? A physical quad is:



idealized as $d \to 0$.

So, it can have direction and orientation ⇒ Quadrupole moment can't just be a vector: needs more degrees of freedom.

$$V_{quad} = \frac{1}{4\pi\epsilon_0 r^3} \int d^3r' \, \rho(r') \left[\frac{3}{2} r'^2 \cos^2\tilde{\theta} - r'^2 \right]$$

$$= \frac{1}{4\pi\epsilon_0} \int d^3r' \, \rho(r') \, \frac{1}{2} \left(\frac{3(\vec{r}' \cdot \vec{r})^2}{r^5} - \frac{(r')^2}{r^3} \right)$$

in coordinates:

$$= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{1}{2} \frac{x_i x_j}{r^5} \int d^3r' \, \rho(\vec{r}') \left(3 x_i' x_j' - \delta_{ij} r'^2 \right)$$

... which can be expressed $\displaystyle \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{1}{2} \frac{x_i x_j}{r^5} Q_{ij}$

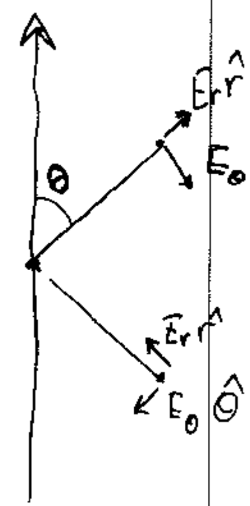
where Q_{ij} is the quadrupole moment tensor:

$$Q_{ij} = \int d^3r' \, \rho(r') \left(3 x_i' x_j' - \delta_{ij} r'^2 \right)$$

Warning: Multipole moments (except for monopole) depend on the origin — a point charge acquires higher moments when moved away from $\vec{r} = 0$.

Dipole fields: Take $\vec{p} = p\hat{z}$.

$$V(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

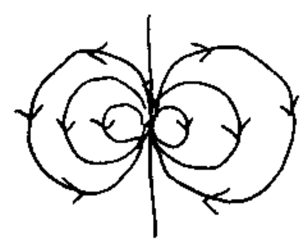


so: $E_r = -\frac{\partial V}{\partial r} = \frac{p\cos\theta}{2\pi\varepsilon_0 r^3}$ $\left(\begin{array}{l}+\text{ for } \theta<\frac{\pi}{2}\\ -\text{ for } \theta>\frac{\pi}{2}\end{array}\right)$

$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$ (Always +)

$E_\phi = 0$. Field lines:

Field is $\sim\frac{1}{r^3}$ dependence, has shape



→ see fig 3.37 for details!

Quad field: