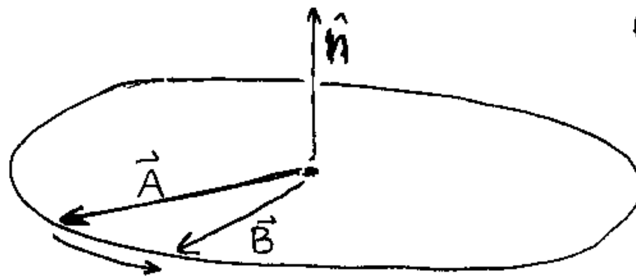
P3310 LECTURE 2

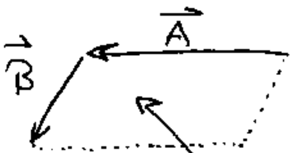
Cross (vector) product:

$$\vec{A}\times\vec{B} = |\vec{A}|\,|\vec{B}|\sin\theta_{AB}\,\hat{n}$$

$\hat{n}$ ← unit vector normal to both $\vec{A}$ and $\vec{B}$.

Right-hand rule:



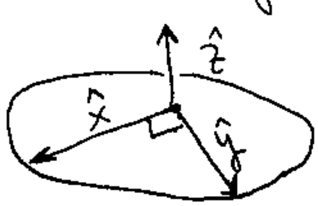


$|\vec{A}\times\vec{B}|$ = area of this parallelogram
($=0$ if $\vec{A}\parallel\vec{B}$
$= AB$ if $\vec{A}\perp\vec{B}$)

Easy way to calculate in components: $\det\begin{pmatrix}\hat{x} & \hat{y} & \hat{z}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{pmatrix}$

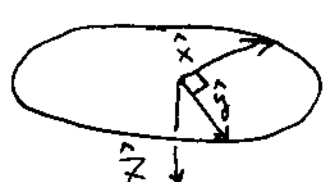
$$= (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z}$$

This is a good place to introduce the concept of <u>cyclic permutations</u>:
Indices can be reordered such that the coordinate system remains right-handed ($\hat{x}\times\hat{y}=\hat{z}$) or becomes left-handed ($\hat{x}\times\hat{y}=-\hat{z}$):



Right-handed

$\hat{x} \Leftrightarrow -\hat{x}$



Left-handed.

Cyclic permutations preserve handedness:

$$x, y, z \rightarrow y, z, x \rightarrow z, x, y$$

(note how the indices "cycle" through.)

Anticyclic permutations reverse handedness:

$$x, z, y \rightarrow z, y, x \rightarrow y, x, z$$

Can also express cross product as:

$$\hat{x} A_y B_z + \text{cyc. perm.} - \text{anticyc. perm.}$$

Other cross product properties:

Distributive: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

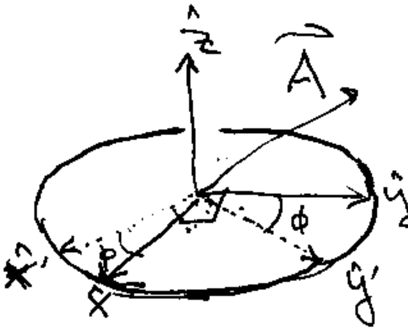
Anticommutative: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ (can see graphically from right-hand rule or from cyclic permutation rule)

Not associative: $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$

Triple product: $\vec{A} \cdot \vec{B} \times \vec{C} = \det \begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{pmatrix}$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A} \cdot \vec{C}) - \vec{C}\,(\vec{A} \cdot \vec{B}) \quad \text{"BAC-CAB" rule}$$

Vector transformation under a change of coordinates:



Rotate coordinate vectors by ϕ about $\hat{z}$:

What are $A_{x'}$, $A_{y'}$?

$$\begin{pmatrix} A_{x'} \\ A_{y'} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

Arbitrary rotations involve 3x3 matrix. Note that matrix elements must satisfy constraint: $U^T U = I$. ("orthogonal" matrix). Check for 2x2 rotation:

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} c^2+s^2 & cs-sc \\ sc-cs & s^2+c^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

Vector calculus:

Introduce gradient vector operator: $\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$

Take a scalar function $f(\vec{r}) = f(x,y,z)$:

Can easily calculate directional derivative $\frac{\partial f}{\partial x}$ along $\hat{x}$, or $\frac{\partial f}{\partial y}$ along $\hat{y}$, etc... but what is derivative along an arbitrary unit vector $\hat{n}$? Remember $\hat{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$!

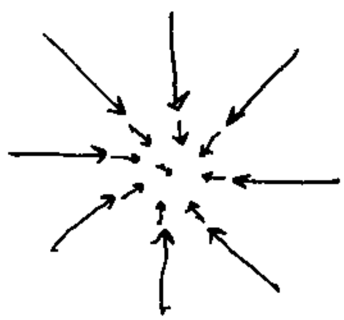
So derivative along $\hat{n}$ is $n_x\frac{\partial f}{\partial x} + n_y\frac{\partial f}{\partial y} + n_z\frac{\partial f}{\partial z} = \hat{n}\cdot\vec{\nabla}f$

Note that $\vec{\nabla}f$ is a vector function of x,y,z $\Rightarrow$ a vector field!
$\hookrightarrow$ "grad f"

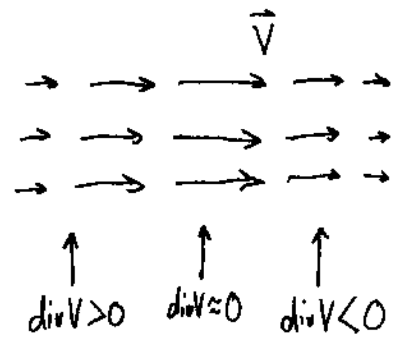
- Gradient's direction is the direction in which f changes most steeply
- $|\vec{\nabla} f|$ is the directional derivative in that direction.

Divergence: $\quad \text{div}\, \vec{V}(\vec{r}) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \vec{\nabla} \cdot \vec{V}(\vec{r})$

- <u>Scalar</u> → Divergence does <u>not</u> have a direction!
- Describes the "spreading out" of the vector field $\vec{V}(\vec{r})$.



This field has large negative divergence



The divergence theorem (very important!)

Take a vector field $\vec{V}(\vec{r})$, and a volume $\mathbb{V}$ enclosed by surface S:

$$\int_{\mathbb{V}} d^3r \; \text{div}\,\vec{V}(r) = \int_{S} \vec{V} \cdot d\vec{A}$$

where $d\vec{A}$ is the unit area element normal to surface S.

Notation: d^3r means volume integral: $dx\,dy\,dz$ or equivalent.
(Griffiths uses "$d\tau$" which is bizarre.)

Amazing result! The <u>volume</u> integral of <u>scalar</u> div V completely determines the <u>surface</u> integral of $\vec{V}$, and vice versa. In electrostatics, $\text{div}\,\vec{E} \leftrightarrow \rho$: divergence of electric field is the charge density.