Multipole expansions.

Consider the potential due to a localized charge distribution, in the region outside the charge. Sep. of variables told us that we can express the potential as:

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

But since we're interested in the region outside the charge, can assume all $A_l = 0$. So

$$V(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta).$$

Examine this expression:

$$V(r,\theta) = \frac{B_0}{r} + \frac{B_1}{r^2}\cos\theta + \frac{B_2}{r^3}P_2(\cos\theta) + \frac{B_3}{r^4}P_3(\cos\theta) + \cdots$$

This is an expansion in powers of r. So at larger and larger r, fewer and fewer terms contribute significantly! At large enough r, only the first nonzero B_l will contribute significantly. The terms have names: $(4\pi\epsilon_0)B_0$: Monopole moment

$(4\pi\epsilon_0)B_1$: Dipole moment

: B_2 : Quadrupole moment

: B_3 : Octopole moment

This is useful as a calculation/approximation technique, but these multipole moments have nice physical interpretations.

How to find the multipole moments? Several ways...

1) We know the potential $V(r_0, \theta)$ at some radius outside the charge distribution. We've already solved this problem! Simply expand $V(r_0,\theta)$ in $P_\ell(\cos\theta)$, and match up the right $\frac{1}{r^{\ell+1}}$ dependence:

$$B_\ell = \frac{(2\ell+1)}{2}\, r_0^{\ell+1} \int_0^\pi V(r_0,\theta) P_\ell(\cos\theta) \sin\theta \, d\theta$$

As noted before, general integral is ugly — but if you are only concerned with large-r behavior the first few terms should be sufficient. Note in particular:

$$\text{Monopole moment} = B_0 = \frac{r_0}{2} \int_0^\pi d\theta \; V(r_0,\theta) \sin\theta$$

$$= \frac{1}{r_0} \cdot (\text{Average potential on } r_0 \text{ sphere})$$

$$\text{Dipole moment} = B_1 = \frac{3r_0^2}{2} \int_0^\pi d\theta \; \sin\theta\cos\theta \; V(r_0,\theta)$$

$$= \frac{3r_0^2}{4} \int_0^\pi d\theta \; \sin 2\theta \; V(r_0,\theta)$$

Note that monopole potential is that of a point charge $Q = 4\pi\varepsilon_0 B_0$. Several ways to show this is equal to total system charge: One is to consider Gauss's Law: Take sphere of radius r as surface:

$$\frac{1}{\varepsilon_0} Q_{enc} = \oint_s \vec{E} \cdot d\vec{A}$$

Now, $\oint \vec{E}\cdot d\vec{A} = \oint E_r\, dA$ where $E_r = \frac{\partial V}{\partial r}$.

So $E_r = \frac{-\partial V}{\partial r} = -\sum_{\ell=0}^{\infty} B_\ell \left[+(\ell+1)\right] r^{-(\ell+2)} P_\ell(\cos\theta)$

and $\oint E_r\, dA = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, r^2 \sin\theta\, E_r(\theta)$

$$= +2\pi \sum_\ell B_\ell r^{-\ell}(\ell+1) \int_0^{\pi} d\theta \sin\theta P_\ell(\cos\theta)$$

In large r limit, this vanishes for $\ell > 0$. (actually the θ integral vanishes anyway, but it's less obvious – think orthogonality)

So $\oint E_r\, dA = -2\pi B_0(-2) = 4\pi B_0 = \frac{1}{\varepsilon_0} Q_{enc}$

$\Rightarrow Q_{enc} = 4\pi\varepsilon_0 B_0$.

Now, consider different scenario. You know the charge distribution $\rho(\vec{r})$, not $V(r_0, \theta)$. How to calculate the multipole moments? Know exact solution:

$V(r) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{\mathscr{r}}$ ... which in general is hard to evaluate.

Consider $\mathscr{r}$: $\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\theta'$

$$= r^2\left(1 + \frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta'\right)$$

If we are far from the charge distribution, then r is large compared to any r' where $\rho \neq 0$. So $\mathscr{r}^2 = r^2(1+\varepsilon)$

$\Rightarrow \frac{1}{\mathscr{r}} = \frac{1}{r}(1+\varepsilon)^{-\frac{1}{2}}$ Binomial expansion: $\frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \dots\right)$

Series coefficient for ε^n is $\binom{1/2}{n} = \frac{\Gamma(\frac{3}{2})}{\Gamma(n+\frac{1}{2})\, n!}$ (I think)

...but for first few it's probably easier just to remember the coefficients: now substituting for ε:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta'\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta'\right)^3 + \dots\right]$$

Now, collect powers of $\frac{r'}{r}$:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 + \frac{r'}{r}(\cos\theta') + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\theta' - 1}{2}\right) \dots\right.$$

$$= \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta)$$

Griffiths says this is surprising — but sep of variables told you it had to be this way! $\left(\frac{1}{r}\right)^l$ dependence needs a $P_l(\cos\theta)$ factor to satisfy $\nabla^2 V = 0$.