Established Friday that the separable solutions to Laplace eqn in spherical coordinates are:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta)$$

Where $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=(3x^2-1)/2$

$P_l(\cos\theta)$ forms a complete, orthogonal set of functions: This means you can form any function $f(\theta)$ from an appropriate sum of the $P_l(\cos\theta)$:

$$\int_{-1}^{1} dx\, P_l^2(x) = \frac{2}{2l+1}$$

and $\int_{-1}^{1} dx\, P_l(x)P_m(x) = 0$ if $l \neq m \Rightarrow \int_{-1}^{1} dx\, P_l(x)P_m(x) = \frac{2}{2l+1}\delta_{lm}$

So, if we assume $f(\theta) = \sum_l C_l P_l(\cos\theta)$,

then $\int_{1}^{-1} d(\cos\theta)\, f(\theta)P_l(\cos\theta)$

$$= \sum_{m=0}^{\infty} C_m \underbrace{\int_{1}^{-1} d(\cos\theta)\, P_l(\cos\theta)\, P_m(\cos\theta)}_{=\frac{-2}{2l+1}\delta_{ml}} = -C_l\,\frac{2}{2l+1}$$

So $C_l = \frac{2l+1}{2}\int_{-1}^{+1} d(\cos\theta)\, f(\theta)P_l(\cos\theta) = +\frac{2l+1}{2}\int_{\pi}^{0}(-\sin\theta)d\theta\, f(\theta)P_l(\cos\theta)$

$$\boxed{C_l = \frac{2l+1}{2}\int_{0}^{\pi} d\theta\, f(\theta)\sin\theta\, P_l(\cos\theta)}$$

So if, for example, we have boundary $V(r_0, \theta) \equiv V_0(\theta)$ fixed on a sphere $R_0$ and we are trying to solve for the potential <u>outside</u>, we know the solution is

$$V(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$$

(since $A^l$ must be zero to prevent blowup at $r \to \infty$.)

on boundary:

$$V(r_0,\theta) = \sum_{l=0}^{\infty} B_l r_0^{-(l+1)} P_l(\cos\theta) = V_0(\theta)$$

But know $V_0(\theta) = \sum_l C_l P_l(\cos\theta)$ where

$$C_l = \frac{2l+1}{2}\int_0^{\pi} d\theta\, V_0(\theta) \sin\theta\, P_l(\cos\theta)$$

so $$B_l = r_0^{l+1}\,\frac{2l+1}{2}\int_0^{\pi} d\theta\, V_0(\theta)\, \sin\theta\, P_l(\cos\theta)$$

Usually this is a serious pain to integrate! Often easier to solve by eye: Work an example:

$$V_0(\theta) = K \sin^2\theta$$

Look at Legendre polynomials: $P_2(\cos\theta) = \frac{1}{2}\left(3\cos^2\theta - 1\right)$, $P_0 = 1$

Note $K\sin^2\theta = K(1-\cos^2\theta)$: $\quad \frac{2}{3}P_2 = \cos^2\theta - \frac{1}{3}$

so $\cos^2\theta = \frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta)$

$$\Longrightarrow K\sin^2\theta = K\left[P_0(\cos\theta) - \frac{1}{3}P_0(\cos\theta) - \frac{2}{3}P_2(\cos\theta)\right]$$

$$= \frac{2}{3}K\left[\ P_0(\cos\theta) - \ P_2(\cos\theta)\right]$$

$$V(r_0,\theta) = \frac{2}{3}K\,P_0(\cos\theta) - \frac{2}{3}K\,P_2(\cos\theta)$$

$$= \sum_{l=0}^{\infty} B_l\, r_0^{-(l+1)} P_l(\cos\theta)$$

Match up coefficients to find $B_0 r_0^{-1} = \frac{2}{3}K$

$$B_2 r_0^{-3} = \frac{-2}{3}K$$

so $$V(r,\theta) = \frac{2}{3}K\left[\frac{r_0}{r}P_0(\cos\theta) - \left(\frac{r_0}{r}\right)^3 P_2(\cos\theta)\right]$$

$$= \frac{2}{3}K\left[\frac{r_0}{r} - \left(\frac{r_0}{r}\right)^3\left(3\cos^2\theta - 1\right)\right] \quad \text{at all } r \geq r_0.$$

→ Remember the $r^l$ or $r^{-(l+1)}$ dependence!

Practice looking at simple angular functions and seeing how to build them from $P_l(\cos\theta)$. Always first try to express $f(\theta)$ as a polynomial in $\cos\theta$: $f(\theta) = \sum_{m=0}^{n} D_m \cos^m\theta$. You will only need $P_0 \cdots P_n$. If you match up the highest power coefficients, then figure out the remainder ... it will all work out! Remember $P_l$'s are orthogonal, so there's only one answer. However you find it, if it works it's the right answer!

→ Several other good examples in Griffiths.