P3310 LECTURE 17

Separation of variables: spherical coordinates.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

Griffiths assumes azimuthal symmetry*: Nothing depends on ϕ.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0$$

As before, assume separable solution: $V(r,\theta) = R(r)\Theta(\theta)$

Multiply by r^2 (remembering we may be screwing things up at $r=0$):

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0$$

Substitute for V:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right)\Theta(\theta) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \Theta}{\partial\theta}\right)R(r) = 0$$

As in cartesian, now divide by $V = R\Theta$:

$$\underbrace{\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right)}_{=K} + \underbrace{\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \Theta}{\partial\theta}\right)}_{=-K} = 0$$

both constants!

Call $K = l(l+1)$

*In Quantum mechanics you will do very similar problems without ϕ-independent assumption.

Now, have radial and angular equations: partial → ordinary DEs.

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \ell(\ell+1)R$$

$\ell(\ell+1)$ is called a separation constant (like k^2 in Cartesian problem)

Solution: try $r^\ell = R(r)$

$$\frac{d}{dr}\left(r^2 \ell r^{\ell-1}\right) = \ell \frac{d}{dr} r^{\ell+1} = \ell(\ell+1) r^\ell = \ell(\ell+1)R \checkmark$$

try $r^{-(\ell+1)}$:

$$\frac{d}{dr}\left(r^2(-\ell-1)r^{-(\ell+2)}\right) = -(\ell+1)\frac{d}{dr}r^{-\ell} = +\ell(\ell+1)r^{-\ell-1} = \ell(\ell+1)R \checkmark$$

So general solution will be $R(r) = Ar^\ell + Br^{-(\ell+1)}$.

Angular equation: $\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = \left[-\ell(\ell+1)\sin\theta\right]\Theta(\theta)$

This equation is much more complicated to solve. Usually just give the answer! First examine:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -\ell(\ell+1)\Theta(\theta)$$

Keeping $\Theta(\theta)$ well behaved at all values of θ from 0 to π will require $\ell(\ell+1)$ to be an integer — unlike rectangular case where k could be anything real. Solutions are the Legendre polynomials in $\cos\theta$:

$$\Theta(\theta) = P_\ell(\cos\theta) \quad \text{where } \ell = 0, 1, 2 \cdots$$

Note – P_ℓ is a polynomial but the variable $x \to \cos\theta$.

The P_ℓ are found using Rodrigues Formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$$

so $P_0(\cos\theta) = 1$

$P_1(\cos\theta)$: $P_1(x) = x$, so $P_1(\cos\theta) = \cos\theta$

$P_2(x) = (3x^2-1)/2 \Rightarrow P_2(\cos\theta) = (3\cos^2\theta - 1)/2$

$P_3(x) = (5x^3-3x)/2$ etc.

$P_4(x) = (35x^4 - 30x^2 + 3)/8$

As with sin & cos, the $P_\ell(\cos\theta)$ form a complete, orthogonal series, so any boundary conditions can be satisfied.

General solution: $V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell(\cos\theta)$

Some usage notes:

- If one boundary is $r \to \infty$, then $A_\ell = 0$ for all A_ℓ
- If there's nothing special about $r=0$ and it's in the solution space, $B_\ell = 0$.
- Angular part: usually, can see how to express boundary conditions in terms of first few P_ℓ. Otherwise, exploit orthogonality relation.