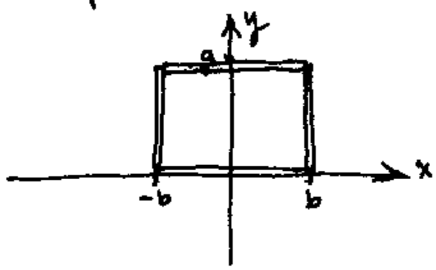
Separation of variables: 2-D example: (say it goes to ∞ in z)



Top and bottom: $V=0$

Left and right: $V = V_0$ constant

Inside, $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

So, know solutions of the form $V_i = V_{ix}(x) V_{iy}(y)$ have behavior

$$\frac{1}{V_x(x)} \frac{d^2 V_x(x)}{dx^2} + \frac{1}{V_y(y)} \frac{d^2 V_y(y)}{dy^2} = 0$$

$\Rightarrow$ both additive terms constant, so

$$\frac{d^2 V_x(x)}{dx^2} = \underbrace{k^2 V_x(x)}_{+\text{ or }-}, \qquad \frac{d^2 V_y(y)}{dy^2} = \underbrace{-k^2 V_y(y)}_{-\text{ or }+}$$

Which do we want to be positive?

+ will have exponential solutions
− will have sin / cos solutions

In the "infinite slot" example last week, choice was obvious because we couldn't have an oscillating solution going to $x = \infty$. Here it is not obvious, so pick one.

$Ae^{+kx} + Be^{-kx}$ is easier to deal with in x since the problem is symmetric about $x=0$.

$$\Rightarrow V_i(x,y) = \left[A e^{kx} + Be^{-kx}\right]\left[C \sin ky + D \cos ky\right]$$

know $V(-b,y) = V(b,y)$ so set $A=B$: $A\left(e^{kx} + e^{-kx}\right)$

Recall $\frac{1}{2}\left(e^{kx} + e^{-kx}\right) = \cosh(kx)$, and absorb constant.

So $V_i(x,y) = \cosh kx\left(C\sin ky + D\cos ky\right)$

Now, look at boundary conditions: $V=0$ at $y=0$, so no $\cos ky$ terms can contribute $\Rightarrow D=0$.

$$V_i(x,y) = C\cosh(kx)\sin(ky)$$

$V=0$ also at $y=a$, so $\sin(ka)=0 \Rightarrow k=\frac{n\pi}{a}$

$$V_n(x,y) = C_n\cosh\frac{n\pi x}{a}\sin\frac{n\pi y}{a}$$ Note → <u>same n</u>!

Now, $V(x,y) = \sum_n V_n(x,y) = \sum_n C_n\cosh\frac{n\pi x}{a}\sin\frac{n\pi y}{a}$

* Before finding C_n: consider completeness and orthogonality of the $\sin\left(\frac{n\pi y}{a}\right)$ series.

<u>Completeness:</u> must be able to construct the given values at boundaries by adding up solutions with appropriate C_n. If our series is <u>complete</u> then we can do this for <u>any</u> function at the boundary:

$$V(b,y) = \sum_n \underbrace{C_n\cosh\frac{n\pi b}{a}}_{\text{constant}\Rightarrow}\underbrace{\sin\frac{n\pi y}{a}}_{\text{fn of } y}$$ must be $=V_0$ at any y from 0 to a.

consider as coeff.

So we need a series of $C_n\cosh\frac{n\pi b}{a} \equiv \tilde{C}_n$ that gives a constant when mult by $\sin\frac{n\pi y}{a}$:

$$\sum_n \tilde{C}_n\sin\frac{n\pi y}{a} = V_0$$

Fourier series is known to be complete, and constants are:

$$\tilde{C}_n = \frac{2}{a}\int_0^a dy\, V_0\sin\frac{n\pi y}{a} \Rightarrow$$

$$\tilde{C}_n = \frac{2V_0}{a}\left(\frac{a}{n\pi}\cos\frac{n\pi y}{a}\right)_a^0$$

$$= \frac{4V_0}{n\pi} \Rightarrow C_n = \tilde{C}_n\left(\cosh\frac{n\pi b}{a}\right)^{-1} \quad \text{for odd } n$$

$$= 0 \quad \text{for even } n$$

$$\Longrightarrow V(x,y) = \frac{4V_0}{\pi}\sum_{n\ \text{odd}}\frac{1}{n}\frac{1}{\cosh\left(\frac{n\pi b}{a}\right)}\cosh\frac{n\pi x}{a}\sin\frac{n\pi y}{a}$$