P3310 LECTURE 14

Another method for solving Laplace eqn: separation of variables. (Technique is also used in quantum mechanics.)

Basic idea: Look for solutions of the form

$$V = X(x)\,Y(y)\,Z(z)$$
$$R(r)\,\Theta(\theta)\,\Phi(\phi)$$
$$R(r)\,\Phi(\phi)\,Z(z)$$

...then take combinations of these solutions (recalling superposition principle) to build a solution that matches the boundary conditions.

Here's the trick:

$$\nabla^2 V = 0 = \nabla^2\left[X(x)\,Y(y)\,Z(z)\right]$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0$$

Note: NOT partial derivs!

Now, divide by XYZ:

$$\underbrace{\frac{1}{X(x)}\frac{d^2X}{dx^2}}_{\text{pure function of } x} + \underbrace{\frac{1}{Y(y)}\frac{d^2Y}{dy^2}}_{\text{pure function of } y} + \underbrace{\frac{1}{Z(z)}\frac{d^2Z}{dz^2}}_{\text{pure function of } z} = 0$$

so this is $f_1(x) + f_2(y) + f_3(z) = 0$. Since f_1, f_2, f_3 are all different functions, they must all be <u>constant</u> for them to add to zero for any value of (x, y, z):

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = C_1 \qquad \frac{1}{Y(y)}\frac{d^2Y}{dy^2} = C_2$$

$C_1 + C_2 + C_3 = 0.$

Now we have independent equations for the three functions:

$$\frac{d^2X(x)}{dx^2} = C_1 X(x)$$

Generally, solutions to these equations are exponentials:

$$X(x) = Ae^{kx} \longrightarrow \text{so} \quad \frac{d^2X}{dx^2} = Ak^2e^{-kx} = k^2e^{-kx}$$

$$C_1 \leftrightarrow k^2.$$

Equation is actually satisfied by sum of exponentials as long as elements of the sum have same k^2:

$$\text{So} \quad X(x) = Ae^{kx} + Be^{-kx} \qquad \text{where } A, B \text{ are constants.}$$

if $C_1 > 0$, exponent real as above.

if $C_1 < 0$, exponent imaginary: $X(x) = Ae^{ikx} + Be^{-ikx}$ (here, $C_1 = -k^2$)

Solutions don't have to be complex $\longrightarrow$ if $A=B$, then solution is $2A\cos kx$

if $A=-B$ then solution is $2Ai\sin kx$

$\longrightarrow$ make A, B imaginary to create real sine.

So most general real solution if $C_1 < 0$ is $C\cos kx + D\sin kx$ where C, D are arbitrary constants.

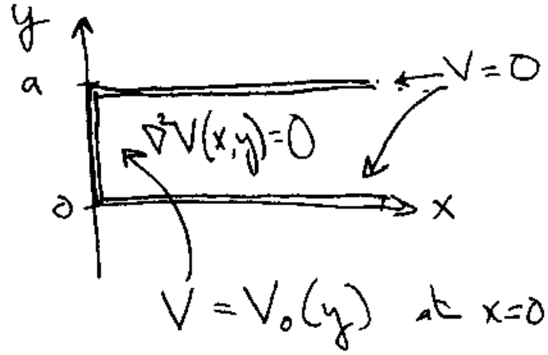
A nice case in particular is 2 dimensions: (or 3 where one is constant):

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \Rightarrow C_1 = -C_2 = k^2$$

So $V_k(x,y) = \left(Ae^{kx} + Be^{-kx}\right)\left(C\cos ky + D\sin ky\right)$ and recall that any sum of these also satisfies Laplace: $V = \sum_k \left[G_k V_k(x,y)\right]$

(and you could switch x, y.)

2-D example in Griffiths:



Boundary conditions:

$y = a$, $y = 0$: $V = 0$

$x = \infty$: $V = 0$

$x = 0$: $V = V_0(y)$

Which solutions can possibly match boundary conditions?

$$\left(Ae^{kx} + Be^{-kx}\right)\left(C\cos ky + D\sin ky\right)$$

Blows up at $x \to \infty$ $\Rightarrow A = 0$

Doesn't vanish at $y=0$ $\Rightarrow C = 0$

Must vanish at $y = a$ $\Rightarrow k = \frac{n\pi}{a}$ where n integer.

If $V_0(y)$ happens to be some multiple of $\sin\frac{n\pi y}{a}$, then

$$V(x,y) = B e^{-\frac{n\pi x}{a}} \sin\frac{n\pi y}{a} \quad \text{(absorbing } D \text{ into } B\text{)}.$$

What about general case of $V_0(y) \neq B\sin\frac{n\pi y}{a}$?

We have one more trick: the separable solution we found can be for any integer n, or a linear combination as above:

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\frac{n\pi y}{a}$$

Satisfies Laplace, plus the upper & lower boundary conditions.

It is the unique solution if

$$V(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi y}{a} = V_0(y) \quad \text{for } 0<y<a.$$

This is just a Fourier expansion of $V_0(y)$, which works for any reasonable function $V_0(y)$:

C_n can be found as follows: Multiply by $\sin\frac{n'\pi y}{a}$ and integrate:

$$\int_0^a dy\, V_0(y)\sin\frac{n'\pi y}{a} = \int_0^a dy\left(\sum_{n=1}^{\infty} C_n \sin\frac{n\pi y}{a}\sin\frac{n'\pi y}{a}\right)$$

Pull out the sum:

$$\int_0^a dy\, V_0(y)\sin\frac{n'\pi y}{a} = \sum_{n=1}^{\infty} C_n \underbrace{\int_0^a dy\,\sin\frac{n\pi y}{a}\sin\frac{n'\pi y}{a}}$$

this is 0 if $n' \neq n$
and $\frac{a}{2}$ if $n' = n$.

So the only contributing term is $n' = n$:

$$\int_0^a dy\, V_0(y)\sin\frac{n'\pi y}{a} = C_{n'}\frac{a}{2}$$

$$\Longrightarrow C_n = \frac{2}{a}\int_0^a dy\, V_0(y)\sin\frac{n\pi y}{a}.$$

Plug this in for C_n in $V(x,y)$ for complete solution.