P3310 LECTURE 13

Image method: Field and potential from a point charge above a grounded conducting plane equal those from the initial point charge plus a fictitious opposite charge at the mirror-image location:

$$V(\vec{r}, z>0) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{-Q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

What if there's something more complicated than a point charge?
→ Solutions to Poisson equations obey superposition, so can add image of entire distribution:

$$V(\vec{r}, z>0) = \frac{1}{4\pi\varepsilon_0}\int_{z'>0} d^3r' \left(\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \frac{-\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'_{mirror}|}\right)$$

where $\vec{r}\,'_{mirror} \equiv (r'_x, r'_y, -r'_z)$

Note - integral above is only over $z'>0$ to avoid any actual charge that might be below the conductor!

Work and energy in image problems: Easy to be fooled!
In $z>0$ region, the fields & forces look exactly like a 2-charge problem with $+Q$ at $z=d$, $-Q$ at $z=-d$. But in that problem work (total energy) is:

$$W = \frac{1}{2}\sum_i q_i V(\vec{r}_i) = \frac{1}{8\pi\varepsilon_0}\left[Q\frac{-Q}{2d} + (-Q)\frac{Q}{2d}\right]$$

$$= \frac{-1}{8\pi\varepsilon_0}\frac{2Q^2}{2d}$$

...but this is twice the energy of the Q + conductor system.

Why? Two ways to see this:

(1) The image charge $-Q$ does not exist! The energy calculation was for the work on two point charges. We only did work on one, so by symmetry the answer is half the energy in the two-charge problem:

$$W = \frac{-1}{8\pi\varepsilon_0}\frac{Q^2}{2d}$$

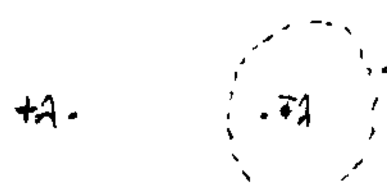
(When we brought charge Q in, its field caused the surface charge on the conductor to change. Why didn't that involve work?)

(2) Energy is stored in the field: $W = \frac{1}{2}\varepsilon_0 \int d^3r\, E^2$.

Since in 2-charge problem $|E_{+z}| = |E_{-z}|$, we know $\int_{z>0} d^3r\, E^2 = \int_{z<0} d^3r\, E^2$, so half the energy is below xy-plane $\rightarrow$ there, $E = 0$ in the actual problem.

Images are only useful in situations where the geometry is especially simple: planes, cylinders, spheres.

Recall last homework assignment: two line charges, parallel & opposite:



Equipotential surface, $V = V_0$ is a cylinder.

In region outside cylinder, this looks exactly like potential of a single charge $+\lambda$ and a conducting cylinder held at $V = V_0$.

Images with spherical conductor: read page 125